Homework #2.

#2.

Let $\vec{u},\vec{v}\in\mathbb{R}^n$. Prove the parallelogram law:

$$\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2=2(\|\vec{u}\|^2+\|\vec{v}\|^2).$$

proof:

$$\|\vec{u}+\vec{v}\|^2+\|\vec{u}-v\|^2=\|\vec{u}\|^2+2\vec{u}\cdot\vec{v}+\|\vec{v}\|^2+\|\vec{u}\|^2-2\vec{u}\cdot\vec{v}+\|\vec{v}\|^2$$
$$=2(\|\vec{u}\|^2+\|\vec{v}\|^2).$$

$\blacksquare$

#3.

Suppose $A\in\mathbb{R}^{n\times n}$ satisfies $\|A\|_2\leq 1$. Prove that $A-\sqrt{2}I$ is invertible.

proof:

Let $x\in \mathrm{Null}(A-\sqrt{2}I)$. Then,

$$A\vec{x}=\sqrt{2}\vec{x}.$$

Therefore, if $\vec{x}\neq 0$ it follows that

$$\frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2}=\sqrt{2}>1,$$

which is a contradiction. Therefore $\vec{x}=0$.

$\blacksquare$

#4

Suppose D is diagonal with entries $d_1,\ldots,d_n\in\mathbb{R}^n$. Prove that

$$\|D\|_p=\max_{1\leq i\leq n}|d_i|.$$

proof:

Let $\vec{x}\in\mathbb{R}^n$ satisfy $\|\vec{x}\|_p=1$. Then,

$$\|D\vec{x}\|_p=\left(\sum_{i=1}^n|x_i d_i|^p\right)^{1/p}\leq\left(\sum_{i=1}^n\max_{1\leq j\leq n}|d_j|^p\cdot|x_i|^p\right)^{1/p}=\max_{1\leq j\leq n}|x_j|\left(\sum_{i=1}^n|x_i|^p\right)^{1/p}$$

$$\Rightarrow\|D\vec{x}\|_p\leq\max_{1\leq j\leq n}|x_j|. \quad *$$

Without loss of generality assume that $\max_{1\leq j\leq n}|d_j|=|d_1|$. Therefore,

$$\|D\vec{e}_1\|_p=|d_1|,$$

and thus by $*$ $\|D\|_p=\max_{1\leq j\leq n}|d_j|$.

$\blacksquare$

#2.6

Solution:

Let $A = I + uv^*$. Then,

$$A(I+\alpha uv^*) = I + uv^* + \alpha uv^* + \alpha uv^* uv^*$$
$$= I + (1+\alpha+\alpha v^* u)uv^*.$$

Therefore, $I + \alpha uv^* = A^{-1}$ if and only if $\alpha = -\frac{1}{1+v^*u}$. It follows that A is singular if $v^*u = -1$. Now, suppose $v^*u=-1$ and let $\vec{x} \in \text{Null}(A)$. Then,

$$A\vec{x} = \vec{x} + uv^*\vec{x} = 0$$
$$\Rightarrow \vec{x} = -(v^*x)\vec{u}$$

Therefore, $\text{Null}(A) = \text{span}(\vec{u})$.

#3.2

Let $\vec{v}$ be an eigenvector of A with corresponding eigenvalue λ. Then,

$$\frac{\|A\vec{v}\|}{\|\vec{v}\|} = \frac{\lambda\|\vec{v}\|}{\|\vec{v}\|} = \lambda \leq \max_{\vec{x}} \frac{\|A\vec{x}\|}{\|\vec{x}\|} = \|A\|.$$

#3.3

a.) $\|\vec{x}\|_2 = \left[\sum_{i=1}^{m} x_i^2\right]^{1/2} \geq \left[\max_{1\leq j\leq n} |x_i|^2\right]^{1/2} = \|\vec{x}\|_\infty.$

b.) $\|\vec{x}\|_2 = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2} \leq \left[\sum_{i=1}^{m} \max_{1\leq j\leq n} |x_i|^2\right]^{1/2} = \sqrt{m}\|\vec{x}\|_\infty.$

c.) Let $\vec{y} \in \mathbb{R}^n$ satisfy $\frac{\|A\vec{y}\|_\infty}{\|\vec{y}\|_\infty} = \|A\|_\infty$. Then,

$$\|A\|_\infty = \frac{\|A\vec{y}\|_\infty}{\|\vec{y}\|_\infty} \leq \frac{\|A\vec{y}\|_2}{\|\vec{y}\|_2/\sqrt{m}} = \sqrt{m}\frac{\|A\vec{y}\|_2}{\|\vec{y}\|_2} \leq \sqrt{m}\|A\|_2.$$

d.) Let $\vec{y} \in \mathbb{R}^n$ satisfy $\frac{\|A\vec{y}\|_2}{\|\vec{y}\|_2} = \|A\|_2$. Then,

$$\|A\|_2 = \frac{\|A\vec{y}\|_2}{\|\vec{y}\|_2} \leq \sqrt{m}\frac{\|A\vec{y}\|_\infty}{\|\vec{y}\|_\infty} \leq \|A\|_\infty.$$

#2.5.

a.) Let $S \in \mathbb{C}^{m\times m}$ be skew-Hermitian. Suppose $\vec{v} \in \mathbb{C}^{m\times m}$ is an eigenvector of S with eigenvalue λ. Then,

$$\lambda \|\vec{v}\|^2 = \langle S\vec{v}, \vec{v}\rangle = \langle \vec{v}, S^*\vec{v}\rangle = \langle \vec{v}, -S\vec{v}\rangle = -\lambda^* \|\vec{v}\|^2$$

$\Rightarrow \lambda = -\lambda^*$

$\Rightarrow \lambda$ is pure imaginary.

b.) Suppose $I-S$ is singular and let $\vec{x} \in \text{Null}(I-S)$ satisfy $\vec{x} \neq 0$. Then,

$$S\vec{x} = \vec{x},$$

which implies 1 is an eigenvalue of S, a contradiction.

c.) Compute:

$$\begin{aligned}
[(I+S)^{-1}\cdot(I+S)]\,[(I-S)^{-1}\cdot(I+S)]^* &= (I-S)^{-1}(I+S)(I-S)(I+S)^{-1} \\
&= (I-S)^{-1}(I-S^2)(I+S)^{-1} \\
&= (I-S)^{-1}(I-S)(I+S)(I+S)^{-1} \\
&= I.
\end{aligned}$$