Homework #7

#1.

Let $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $H = (\operatorname{span}\{\vec{v}\})^{\perp}$.

- Find $P_{\vec{v}}$
- Find P_H
- Find Q_H

Solution:

- $P_{\vec{v}} = \dfrac{\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}} = \dfrac{1}{6}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} = \dfrac{1}{6}\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$

$$\Rightarrow P_{\vec{v}} = \begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 1/6 & -1/6 \\ 1/3 & -1/6 & 1/6 \end{bmatrix}$$

- $P_H = I - P_{\vec{v}}$

$$\Rightarrow P_H = \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & -1/6 & 1/6 \\ -1/3 & 1/6 & -1/6 \end{bmatrix}$$

- $Q_H = I - 2P_{\vec{v}}$

$$\Rightarrow Q_H = \begin{bmatrix} -1/3 & 2/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{bmatrix}$$

$\blacksquare$

#2.

Let A be an $m \times n$ matrix. Prove that A has full rank if and only if the diagonal matrices of R are nonzero.

proof:

Let $A \in \mathbb{R}^{m \times n}$ with reduced QR factorization $A = \hat{Q}\hat{R}$. Since the colums of $\hat{Q}$ are orthonormal it follows that $\hat{Q}$ is full rank. Consequently, $\operatorname{rank}(A) = \operatorname{rank}(\hat{R})$. Now,

$$\operatorname{rank}(\hat{R}) < n \Leftrightarrow \det(\hat{R}) = \prod_{j=0} r_{jj} = 0.$$

$\blacksquare$

#7.3

Let $A \in \mathbb{R}^{n \times n}$, Prove,

$$|\det(A)| \leq \prod_{j=1}^{n} \|a_j\|_2$$

and give a geometric interpreation.

proof:

Let $A \in \mathbb{R}^{n \times n}$ with corresponding QR decomposition. Then,

$$|\det(A)| = |\det(Q)| \cdot |\det(R)|$$
$$= \prod_{j=1}^{n} |r_{jj}|.$$

Now, $|r_{jj}| = \|P_j a_j\|_2$, where P_j is the j-th orthogonal projection in Gram-Schmi

Since $\|P_j\|_2 = 1$ it follows that $|r_{jj}| < \|a_j\|_2$.

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