Lecture 1: Matrix-Vector Multiplication

Let $\vec{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ $(\vec{x} \in \mathbb{C}^n, A \in \mathbb{C}^{n\times m})$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The vector $\vec{b} \in \mathbb{R}^m$ defined by

$$b_i = \sum_{j=1}^{n} a_{ij} x_j,$$

is called the matrix-vector product: $A\vec{x} = \vec{b}$.

Linearity:

As a map $\vec{x} \mapsto A\vec{x}$ is linear:

$$A(\alpha\vec{x} + \vec{y}) = \alpha A\vec{x} + A\vec{y},$$

for any $\vec{x}, \vec{y} \in \mathbb{C}^n$, $\alpha \in \mathbb{C}$.

Multiplication Revisted:

$$\vec{b} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$\Rightarrow$ $\vec{b}$ is a linear combination of the colums of A!

$\vec{x}$ acts on A to produce $\vec{b}$

Let $\vec{a}_j$ denote j-th column of A:

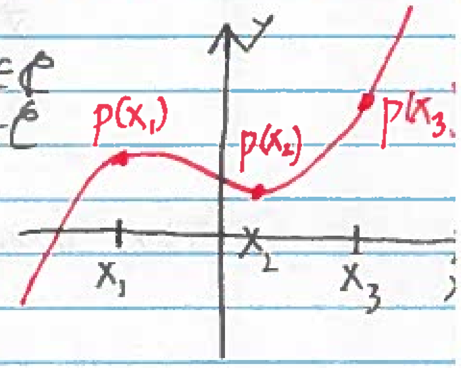
$$\vec{b} = \sum_j x_j \vec{a}_j.$$

example (Vandermonde):

Let $\{x_1, x_2, \ldots, x_m\}$ be a sequence of numbers.

Define

$$p(x) = c_0 + c_1 x + \ldots + c_{n-1} x^{n-1}, \quad p_i \in \mathbb{C}$$
$$q(x) = d_0 + d_1 x + \ldots + d_{n-1} x^{n-1}, \quad q_i \in \mathbb{C}$$



$$(p+q)(x_i) = p(x_i) + q(x_i)$$
$$(\alpha p)(x_i) = \alpha p(x_i)$$

$\downarrow$

The map from coefficients c_j to sampled values $p(x_i)$ is linear.

What is the matrix of this map?

$$A\vec{c} = \vec{p} = \{p(x_1), p(x_2), \ldots, p(x_{n-1})\}.$$

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

$$\Rightarrow A\vec{c} = c_0 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} + \ldots + c_{n-1} \begin{bmatrix} x_1^{n-1} \\ x_2^{n-1} \\ \vdots \\ x_m^{n-1} \end{bmatrix}.$$

example (Differentiation):

Let $p(x) = c_0 + c_1 x + \dots + c_{n-1} x^{n-1}$. The map from coefficients, to coeffients of $p'(x)$ is linear

$$A\vec{c} = A \cdot \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_2 \\ \vdots \\ (n-1)c_{n-1} \\ 0 \end{bmatrix} = c_0 \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + c_{n-1} \cdot \begin{bmatrix} 0 \\ 0 \\ \vdots \\ (n-1) \\ 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & n-1 \\ 0 & 0 & 0 & & 0 \end{bmatrix}$$

example (Integration):

Let $p(x) = c_0 + c_1 x + \dots + c_{n-1} x^{n-1}$. The map from coefficients, to coefficients of $\int_0^x p(s)\,ds$ is linear.

$$A\vec{c} = \begin{bmatrix} 0 \\ c_0 \\ c_1/2 \\ c_2/3 \\ \vdots \\ c_{n-1}/n-1 \end{bmatrix} \Rightarrow c_0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1/3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + c_{n-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1/n-1 \end{bmatrix}$$

$$\Rightarrow A = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1/2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/\dots \end{bmatrix}}_{n\text{-colums}} \Big\} n+1 \text{ rows}$$

Matrix times a matrix

$$b_{ij} = \sum_{k=1}^{m} a_{ik} c_{kj}, \quad B = A \cdot C$$

$$\left[\begin{array}{c|c|c|c} \vec{b}_1 & \vec{b}_2 & \cdots & b_n \end{array}\right] = \left[\begin{array}{c|c|c|c} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \end{array}\right] \left[\begin{array}{c|c|c|c} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{array}\right]$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{bmatrix}$$

$$\Rightarrow \vec{b}_1 = c_{11} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + c_{21} \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

$$\vec{b}_2 = c_{12} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + c_{22} \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

The columns of B are linear combinations of the columns of A!

$$\vec{b}_j = \sum_{k=1}^{m} c_{kj} \vec{a}_k$$

Outer Product

Let $u \in \mathbb{R}^n$, $v \in \mathbb{R}^n$.

$$\vec{u} \cdot \vec{v}^T = B$$

$$\vec{v}^T = [v_1 \ v_2 \ \cdots \ v_n]$$

$$\Rightarrow B = \left[\begin{array}{c|c|c|c} v_1\vec{u} & v_2\vec{u} & \cdots & v_n\vec{u} \end{array} \right] = \begin{bmatrix} v_1u_1 & v_2u_1 & \cdots & v_nu_n \\ v_1u_2 & v_2u_2 & \cdots & v_nu_n \\ \vdots & \vdots & & \vdots \\ v_1u_n & v_2u_n & \cdots & v_nu_n \end{bmatrix}$$

Range and Nullspace

$$\text{range}(A) = \{\vec{b} \in \mathbb{R}^m : \exists \vec{x} \in \mathbb{R}^n \text{ and } A\vec{x} = \vec{b}\}$$

Thm: $\text{Range}(A) = \text{span}\{\text{Col}(A)\}$.

proof:

1. Let $\vec{b} \in \text{Range}(A)$. Therefore, there exists $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$. Consequently, if we let let $\vec{a}_i$ denote the i-th column of A it follows that
$$\sum_{i=1}^{n} x_i \vec{a}_i = \vec{b}.$$
Therefore, $\vec{b} \in \text{Col}(A)$ and thus $\text{Range}(A) \subseteq \text{Span}\{\text{Col}(A)\}$

2. Let $\vec{b} \in \text{span}\{\text{Col}(A)\}$. Then, there exists $\{x_1, x_2, \dots, x_n\}$ such that
$$\vec{b} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n.$$
$$\Rightarrow \vec{b} = \sum_{i=1}^{n} x_i \vec{a}_i$$
$$= A\vec{x}.$$
Therefore, $\vec{b} \in \text{Range}(A)$ and thus $\text{span}\{\text{Col}(A)\} \subseteq \text{Range}(A)$.

Def:

$$\text{null}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = 0\}.$$
$$\text{rank}(A) = \dim(\text{Range}(A)).$$

A matrix is full rank if
$$\text{rank}(A) = \min\{m, n\}.$$

Thm: A matrix $A \in \mathbb{R}^{m \times n}$ with $m \geq n$ has full rank if only if A is one-to-one.

proof

1. Suppose A is full rank. Then $\dim(\text{Col}(A)) = n$ and hence the columns of A are linearly independent. Consequently, the columns of A form a basis. Now suppose for some $b \in \mathbb{R}^m$ there exists $\vec{x}, \vec{y} \in \mathbb{R}^n$ such that

$$A\vec{x} = \vec{b} \quad \text{and} \quad A\vec{y} = b$$
$$\Rightarrow \sum x_i \vec{a}_i = \vec{b} \text{ and } \sum y_i \vec{a}_i = \vec{b}$$

which contradicts linear independence.

2. Suppose A is one-to-one. For contradiction, suppose A is not full rank. Then, the columns of A are linearly dependent. Therefore, there exists $\vec{c} \neq 0$ such that

$$\sum c_i \vec{a}_i = 0.$$
$$\Rightarrow A\vec{c} = 0.$$

However, for any $\vec{x} \in \mathbb{R}^n$, $A\vec{x} = A(\vec{x} + \vec{c})$ which is a contradiction.

$\blacksquare$

Inverse:

$A \in \mathbb{R}^{m\times m}$, with A full rank is called non-singular.

$\Rightarrow$ The columns of A form a basis.

$\Rightarrow$ There exists coefficients z_{ij} such that:

$$e_j = \sum_{i=1}^{m} z_{ij} \vec{a}_i \quad \text{(expand } e_j \text{ in basis)}$$

$$\Rightarrow A Z = I$$

The matrix Z is the inverse of A.

Thm: $A \in \mathbb{R}^{m\times m}$, the following are equivalent.

(a) A is nonsingular

(b) A is full rank

(c) $\text{range}(A) = \mathbb{R}^m$

(d) $\text{null}(A) = \{0\}$.

(e) 0 is not an eigenvalue

(f) $\det(A) \neq 0$.

Now, in words $A^{-1}\vec{b} = \vec{x}$ means $\vec{x}$ solves the equation $A\vec{x} = \vec{b}$. But, this means

$$\sum x_i \vec{a}_i = \vec{b}$$

$\Rightarrow A^{-1}\vec{b}$ is the vector of coefficients of the expansion of $\vec{b}$ in the basis of columns of A!

$\Rightarrow$ Multiplication by A^{-1} is a change of basis!