Lecture 2: Orthogonality

Inner Products

Let $\vec{x}, \vec{y} \in \mathbb{C}^n$, then

$$\vec{x}^* \vec{y} = \langle \vec{x}, \vec{y} \rangle = \sum_{i=1} \overline{x_i}\, y_i,$$

where $\overline{x_i}$ denotes complex conjugate.

1. $\|\vec{x}\| = \langle \vec{x}, \vec{x} \rangle^{1/2}$.
2. $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \cdot \|\vec{y}\| \cos(\theta)$.

Two vectors $\vec{x}, \vec{y}$ are orthogonal if
$$\langle \vec{x}, \vec{y} \rangle = 0.$$

Theorem - The vectors in an orthogonal set S are linearly independant.

~~proof~~:

For contradiction, suppose S is not a linearly independent set. Then, there exists $\vec{v}_k \in S$ and c_i such that

$$\vec{v}_k = \sum_{i \neq k} c_i \vec{v}_i$$

$$\Rightarrow \langle \vec{v}_k, \vec{v}_k \rangle = 0,$$
$$\Rightarrow \vec{v} = 0$$

a contradiction. $\blacksquare$

Components - Let $\{\vec{q}_1, \ldots, \vec{q}_n\}$ be an orthonormal set and let $\vec{v} \in \mathbb{C}^n$.

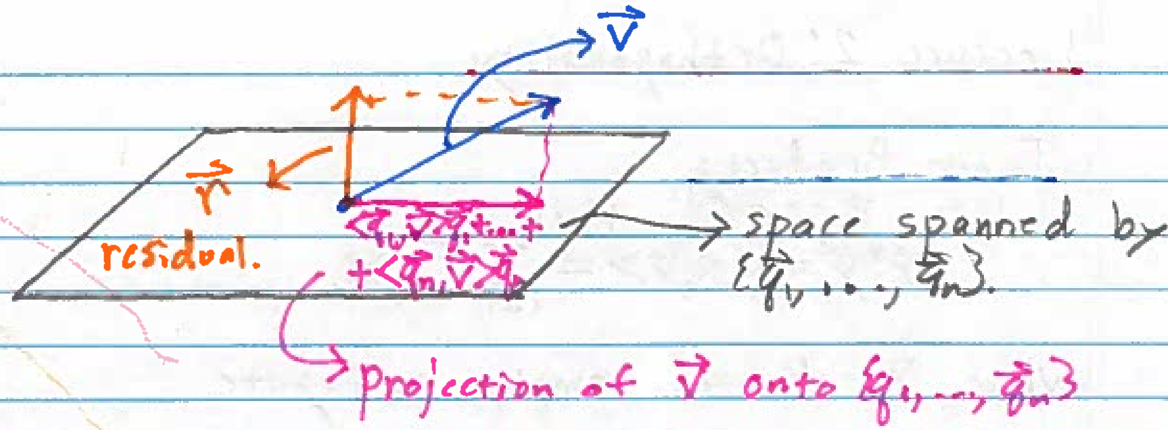
Define

projections onto $\vec{q}_i$

$$\vec{r} = \vec{v} - (\langle \vec{q}_1, \vec{v} \rangle \vec{q}_1 + \ldots \langle \vec{q}_n, \vec{v} \rangle \vec{q}_n)$$

($\vec{r}$ is called the residual)

$$\Rightarrow \langle \vec{q}_i, \vec{r} \rangle = \langle \vec{q}_i, \vec{v} \rangle - \langle q_i, \vec{v} \rangle \langle \vec{q}_i, \vec{q}_i \rangle = 0.$$



If $\{\vec{q}_i\}$ is a basis for $\mathbb{C}^m$, then

$$\vec{v} = \sum_{i=1}^{m} \langle \vec{q}_i, \vec{v}\rangle \vec{q}_i \rightarrow \text{project onto components}$$
$$= \sum_{i=1}^{m} (\vec{q}_i^{\,T} \vec{v}) \vec{q}_i$$
$$= \sum_{i=1}^{m} \vec{q}_i \vec{q}_i^{\,T} \vec{v}$$
$$= \sum_{i=1}^{m} (\underbrace{\vec{q}_i \cdot \vec{q}_i^{\,T}}_{\text{matrices}}) \vec{v}$$

(projections) → sum of rank one matrices.

Unitary Matrices:

A matrix is unitary if $Q^T = Q^{-1}$!

Proposition – The columns of a unitary matrix are orthonormal.

proof:

$$\langle \vec{q}_i, \vec{q}_j\rangle = \langle Q\vec{e}_i, Q\vec{e}_j\rangle$$
$$= (Q\vec{e}_i)^T \cdot Q\vec{e}_j$$
$$= \vec{e}_i^{\,T} Q^T \cdot Q\vec{e}_j$$
$$= \vec{e}_i^{\,T} Q^{-1} Q\vec{e}_j$$
$$= \vec{e}_i^{\,T} \cdot \vec{e}_j$$
$$= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o.w.} \end{cases}$$

Matrices Revisited

In the previous proof I used the fact that the columns of the matrix Q are given by:

$$\vec{q}_i = Q\vec{e}_i.$$

This follows from:

$$Q\vec{x} = Q(x_1\vec{e}_1 + \dots + x_n\vec{e}_n)$$
$$= x_1 Q\vec{e}_1 + \dots + x_n Q\vec{e}_n$$

Result:

$Q\vec{x} \rightarrow$ linear combination of columns of Q.

$Q^*\vec{b} \rightarrow$ vector of coefficients of $\vec{b}$ in the basis of columns of Q.

Unitary Matrices preserve geometry!

$$\langle Q\vec{x}, Q\vec{y}\rangle = (Q\vec{x})^T(Q\vec{y})$$
$$= \vec{x}^T Q^T Q\vec{y}$$
$$= \vec{x}^T\vec{y}$$
$$= \langle\vec{x},\vec{y}\rangle$$