Lecture 3: Norms

A norm $\|\cdot\|: \mathbb{R}^{m\times n} \mapsto \mathbb{R}$ is a function that defines the length or size of a matrix.

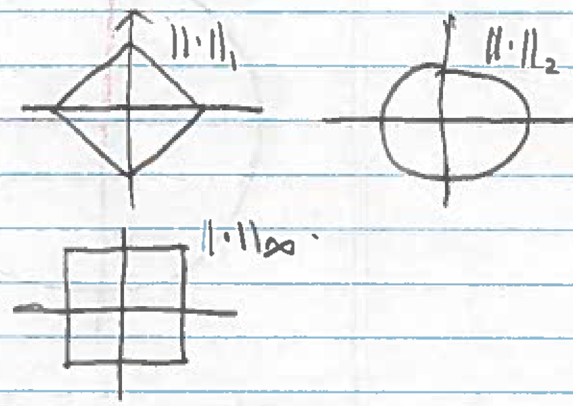
1. $\|A\| \geq 0$, $\|A\| = 0 \iff A = 0$ (positivity)
2. $\|A+B\| \leq \|A\| + \|B\|$ (triangle inequality)
3. $\|\alpha A\| = |\alpha| \cdot \|A\|$. (scale invariant).

Let $\vec{x} \in \mathbb{R}^n$

$$\|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

$$\|\vec{x}\|_\infty = \max_{1\leq i\leq n} |x_i|.$$

$$\|\vec{x}\|_A = \langle \vec{x}, A\vec{x} \rangle$$



Proposition

$$\lim_{p\to\infty} \|\vec{x}\|_p = \|\vec{x}\|_\infty.$$

proof: Let $y = \|\vec{x}\|_p$

$$= \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

$$= \left(\sum_{i=1}^{n} \|\vec{x}\|_\infty^p \frac{|x_i|^p}{\|\vec{x}\|_\infty^p}\right)^{1/p}$$

$$= \|\vec{x}\|_\infty \left(\sum_{i=1}^{n} \frac{|x_i|^p}{\|\vec{x}\|_\infty^p}\right)^{1/p}$$

$$\Rightarrow \ln(y) = \ln(\|\vec{x}\|_\infty) + \frac{1}{p} \ln\left(\sum_{i=1}^{n} \frac{|x_i|^p}{\|\vec{x}\|_\infty^p}\right)$$

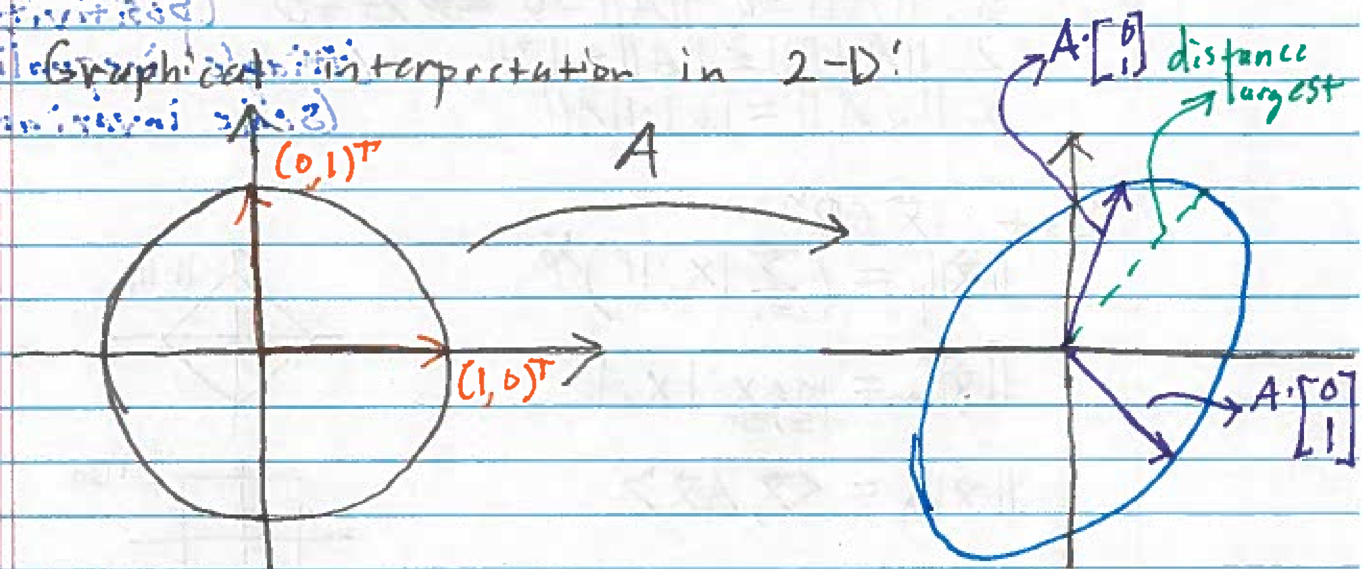
Now, $\frac{|x_i|^p}{\|\vec{x}\|_\infty^p} < 1 \Rightarrow \frac{1}{p} \ln\left(\sum_{i=1}^{n} \frac{|x_i|^p}{\|\vec{x}\|_\infty^p}\right) \leq \frac{1}{p} \ln(n)$

$$\Rightarrow \lim \ln(y) = \ln(\|\vec{x}\|_\infty)$$

Induced Norms:

$$\|A\| = \sup_{\substack{\vec{x}\in\mathbb{R}^n \\ \|\vec{x}\|=1}} \|A\vec{x}\| \quad \text{(measures greatest stretch).}$$

Graphical interpretation in 2-D:



$$\|A\|_1 = \max_{1\le j\le n} \|a_j\|_1 \quad \text{(max column sum).}$$

proof: Let $\vec{x}\in\mathbb{R}^n$ and assume $\|\vec{x}\|_1 = 1$. Then,

$$\begin{aligned}
\|A\vec{x}\|_1 &= \Big\|\sum_{j=1}^{n} x_j \vec{a}_j\Big\|_1 \\
&\le \sum_{j=1}^{n} \|x_j \vec{a}_j\|_1 \\
&\le \sum_{j=1}^{n} |x_j|\cdot\|a_j\|_1 \\
&\le \sum_{j=1}^{n} |x_j|\cdot \max_{1\le i\le n}\|a_i\|_1 \\
&= \max_{1\le i\le n}\|a_i\| \sum_{j=1}^{n} |x_j| \\
&= \max_{1\le i\le n}\|a_j\|
\end{aligned}$$

Choose $\vec{x} = \vec{e}_K$, where this maximum is obtained.

Cauchy-Schwarz.

$$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|_2 \|\vec{y}\|_2$$

Parallelogram Law

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2(\|\vec{u}\|^2 + \|\vec{v}\|^2) \quad \text{(Homework)}.$$

Outer Product.

$\vec{u}, \vec{v} \in \mathbb{R}^n$. Let

$$A = \vec{u}\,\vec{v}^T$$

What is

$\|A\|_2$?

$$\|A\vec{x}\|_2 = \|\vec{u}\,\vec{v}^T\vec{x}\|_2 = \|\vec{u}\|_2 \cdot |\vec{v}^T\vec{x}|_2$$

$$\leq \|\vec{u}\|_2 \cdot \|\vec{v}\|_2 \cdot \|\vec{x}\|_2$$

Can I make equal? Yes, set $\vec{x} = \dfrac{\vec{v}}{\|\vec{v}\|}$. Then,

$$\|A\vec{x}\|_2 = \left\| \vec{u}\,\vec{v}^T \cdot \frac{\vec{v}}{\|\vec{v}\|} \right\| = \|\vec{u} \cdot \|\vec{v}\|_2\| = \|\vec{u}\|_2 \cdot \|\vec{v}\|_2 .$$