Lecture 7: QR Factorization

$A \in \mathbb{R}^{m \times n}$

$A = [\vec{a}_1 | \cdots | \vec{a}_n]$.

Gram-Schmidt:

$$\vec{a}_1 = r_{11}\vec{q}_1$$
$$\vec{a}_2 = r_{12}\vec{q}_1 + r_{22}\vec{q}_2$$
$$\vec{a}_3 = r_{13}\vec{q}_1 + r_{23}\vec{q}_2 + r_{33}\vec{q}_3$$
$$\vdots$$
$$\vec{a}_n = r_{1n}\vec{q}_1 + r_{2n}\vec{q}_2 + \cdots + r_{nn}\vec{q}_n$$

$$A = \underbrace{[\vec{q}_1 | \cdots | \vec{q}_n]}_{m \times n} \underbrace{\left.\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}\right\} n \times n}_{\text{upper triangular.}}$$

$A = \hat{Q}\hat{R}$ (reduced QR)

$\downarrow$

$A = QR$ (Full QR append rows and columns).

Algorithm (Math Version):

$$\vec{q}_1 = \frac{\vec{a}_1}{r_{11}}, \quad r_{ij} = \vec{q}_i^{\,*}\vec{a}_j$$

$$\vec{q}_2 = \frac{\vec{a}_2 - r_{12}\vec{q}_1}{r_{22}} \qquad |r_{jj}| = \|\vec{a}_j - \sum_{i=1}^{j-1} r_{ij}\vec{q}_i\|_2$$

$$\vec{q}_3 = \frac{\vec{a}_3 - r_{13}\vec{q}_1 - r_{23}\vec{q}_2}{r_{33}}$$

$$\vdots$$

$$\vec{q}_n = \frac{\vec{a}_n - \sum_{i=1}^{n-1} r_{in}\vec{q}_i}{r_{nn}}$$

Algorithm (Pseudo Code):

$$
\begin{array}{l}
\text{for } j = 1 \text{ to } n \\
\quad \vec{v}_j = \vec{a}_j \\
\quad \text{for } i = 1 \text{ to } j-1 \\
\qquad r_{ij} = \vec{q}_i \cdot \vec{a}_j \\
\qquad \vec{v}_j = \vec{v}_j - r_{ij}\,\vec{q}_i \\
\quad \text{end} \\
\quad r_{jj} = \|\vec{v}_j\|_2 \\
\quad \vec{q}_2 = \vec{v}_j / r_{22} \\
\text{end.}
\end{array}
$$

Solving Equations by QR.

$$A\vec{x} = \vec{b}$$

$$\downarrow$$

$$QR\vec{x} = \vec{b}$$

$$R\vec{x} = Q^*\vec{b} \quad \text{easy inverse!}$$

Let $\vec{y} = Q^*\vec{b}$

$$\Rightarrow R\vec{x} = \vec{y}$$

Solve $R\vec{x} = \vec{y}$ by substitution.