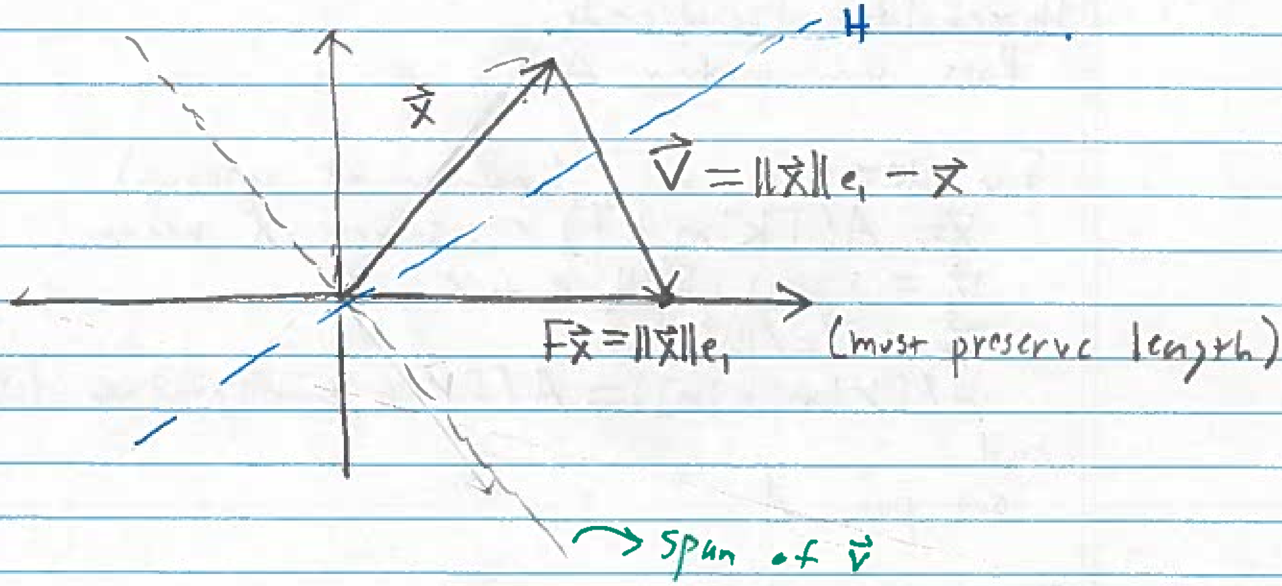
Lecture 10: Householder Triangularization

$$\underbrace{Q_n \cdots Q_2 Q_1}_{Q^*} A = R \xrightarrow[\text{invert } Q^*]{} A = QR$$

$$\begin{bmatrix} * & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \xrightarrow{Q_1 A} \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & \cdot & \cdot \end{bmatrix} \xrightarrow{Q_2 A} \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & 0 & \cdot \end{bmatrix}$$

$$Q_k = \begin{bmatrix} I & 0 \\ 0 & F \end{bmatrix}$$

$I \sim (k-1) \times (k-1)$ identity
$F \sim (m-k+1, m-k+1)$ unitary matrix

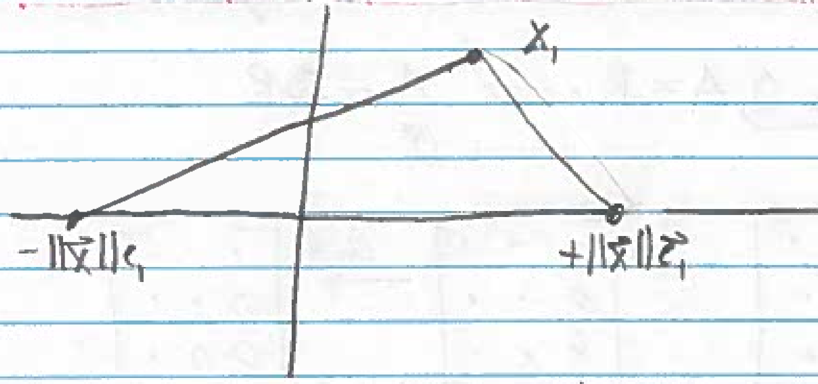


Projection onto H:

$$P\vec y = \left(I - \frac{\vec v \vec v^*}{\vec v^* \vec v}\right)\vec y = \vec y - \vec v\left(\frac{\vec v^* \vec y}{\vec v^* \vec v}\right)$$

$$\Rightarrow F\vec y = \left(I - 2\frac{\vec v \vec v^*}{\vec v^* \vec v}\right)\vec y = \vec y - 2\vec v\left(\frac{\vec v^* \vec y}{\vec v^* \vec v}\right)$$

Another reflector:



choose $v = \text{sign}(x_1)\,\|\vec{x}\|e_1 + x.$

Householder Pseudocode:

Pass $m \times n$ matrix A.

for $k = 1$ to n (number of columns)

$\vec{x} = A([k{:}m, k]) \rightarrow$ select $\vec{x}$ vector

$\vec{v}_k = \text{sign}(x_1)\|x\|_2\, e_1 + x$

$\vec{v}_k = \vec{v}_k / \|\vec{v}_k\|_2$

$A([k{:}m, k{:}n]) = A([k{:}m, k{:}n]) - 2v_k \cdot (v_k^* A([k{:}m, k{:}n]$

end

output A.

Flop Count:

1. $v_k^* A([k{:}m, k{:}n]) = 2(m-k)(n-k)$ $\vec{y}^*$
2. $2v_k \vec{y}^* = (m-k)(n-k)$
3. $A([k{:}n, k{:}n]) - \ldots = (m-k)(n-k)$ subtractions.

$$\text{Flops} \sim \sum_{k=1}^{n} 4(m-k)(n-k) = 4\sum_{k=1}^{n}(mn - k(m+n) + k^2)$$

$$\sim 4mn^2 - \frac{4n^2}{2}(m+n) + \frac{4n^3}{3}$$

$$= 2mn^2 - \frac{2n^3}{3}$$ big difference!!

Calculation of $Q^* b$.

for $k = 1:n$

$\vec{b}_{k:m} = \vec{b}_{k:m} - 2\vec{v}_k(\vec{v}_k^* \vec{b}_{k:m})$

end

Calculation of Qx

for $k = n$ down to 1

$\vec{x}_{k:m} = \vec{x}_{k:m} - 2\vec{v}_k(\vec{v}_k^* \vec{x}_{k:m})$.

How would we actually form Q??

$Q\vec{e}_j$ gives column j of Q!!