MST 386/686
Fall 2021
Exam #1
09/23/21

Name (Print): Key

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

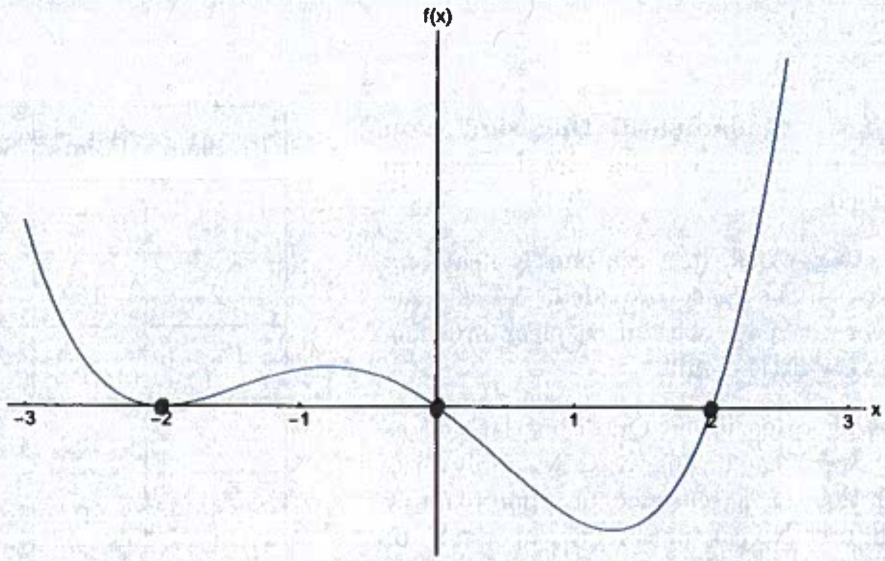
Do not write in the table to the right.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	20	
6	20	
Total:	100	

1. (15 points) Consider the differential equation

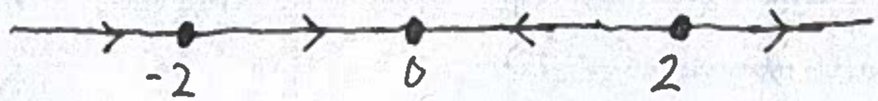
$$\dot{x} = f(x),$$

where $f(x)$ is plotted below.

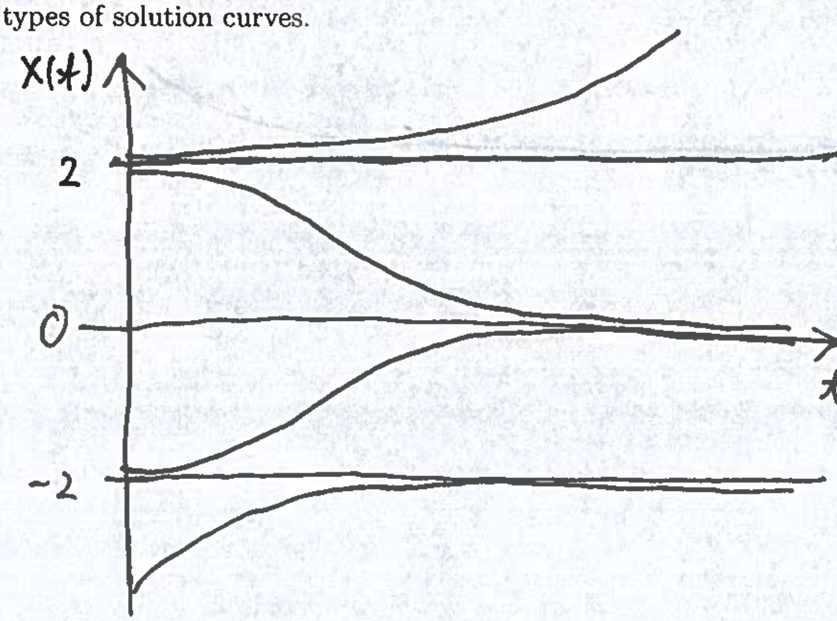


(a) (5 points) **Short Answer:** On the figure indicate any fixed points, i.e. equilibrium points, for this differential equation.

(b) (5 points) **Short Answer:** Sketch a one-dimensional phase portrait for this problem



(c) (5 points) **Short Answer:** On one axis, sketch the corresponding solutions curves $x(t)$ for this problem. Your solution curves should contain all possible qualitatively different types of solution curves.



2. (15 points) Consider the following SIS model with saturating incidence and saturating treatment:

$$\dot{S} = -\frac{\beta IS}{1+\sigma S} + \frac{\alpha I}{1+\gamma I},$$
$$\dot{I} = \frac{\beta IS}{1+\sigma S} - \frac{\alpha I}{1+\gamma I},$$

where $\beta, \sigma, \alpha, \gamma > 0$ are constants.

(a) (5 points) Determine the units of the constants β, σ, α, and γ.

$[\beta] = 1/\text{pop}\cdot\text{time}$ $\quad [\sigma] = 1/\text{pop}$

$[\alpha] = 1/\text{time}$ $\quad [\gamma] = 1/\text{pop.}$

(b) (5 points) Show that the total population $N = S + I$ is constant in time and use this to reduce this system to a single differential equation in I.

$$N = S + I$$
$$\Rightarrow \dot{N} = \dot{S} + \dot{I} = 0$$
$$\Rightarrow \dot{I} = \frac{\beta I(N-I)}{1+\sigma(N-I)} - \frac{\alpha I}{1+\gamma I}$$

(c) (5 points) Through an appropriate dimensionless change of variables show that the system derived in part (b) is equivalent to the following dimensionless equation

$$\frac{dx}{d\tau} = \mathcal{R}_0 \frac{x(1-x)}{1+A(1-x)} - \frac{x}{1+Bx},$$

where $\mathcal{R}_0$, A and B are dimensionless quantities to be determined.

$$\tau = \alpha t, \quad x = I/N$$

$$\Rightarrow \alpha N \frac{dx}{d\tau} = -\frac{\beta N^2 x(1-x)}{1+\sigma N(1-x)} - \frac{\alpha N x}{1+\gamma N x}$$

$$\Rightarrow \frac{dx}{d\tau} = -\frac{\mathcal{R}_0 x(1-x)}{1+A(1-x)} - \frac{x}{1+Bx},$$

Where $\mathcal{R}_0 = \frac{\beta N}{\alpha}$, $A = \sigma N$, $B = \gamma N$.

3. (15 points) The simplest model of malaria assumes that the mosquito population is at equilibrium and models the dynamics of the infected population by

$$\dot{I} = \frac{\alpha\beta I}{\alpha I + r}(N - I) - \mu I,$$

where $\alpha, \beta, r, N, \mu > 0$ are all constants.

(a) (10 points) Determine the fixed points for this problem and analyze their stability. You do not need to and are not expected to nondimensionalize this problem.

$\dot{I}=0 \Rightarrow I=0, \ \dfrac{\alpha\beta(N-I)}{\alpha I+r} - \mu = 0$

$\Rightarrow I=0, \ \alpha\beta N - \alpha\beta I - \mu\alpha I - \mu r = 0$

$\Rightarrow I=0, \ \dfrac{\alpha\beta N - \mu r}{\alpha(\mu+\beta)} = I$

Letting $f(I) = \dfrac{\alpha\beta I}{\alpha I + r}(N-I) - \mu I \Rightarrow \lim\limits_{I\to\infty} f(I) = -\infty$. Therefore, the rightmost fixed point is stable.

$\Rightarrow \dfrac{\alpha\beta N - \mu r}{\alpha(\mu+\beta)}$ is stable if $\alpha\beta N - \mu r < 0$.

is unstable if $\alpha\beta N - \mu r > 0$

(b) (5 points) Determine a basic reproduction number $\mathcal{R}_0$ so that if $\mathcal{R}_0 < 1$ the disease is eliminated.

It follows that the basic reproduction number is given by

$$\mathcal{R}_0 = \frac{\alpha\beta N}{\mu r}$$

4. (15 points) Consider the system of differential equations

$$\frac{dx}{dt} = f(x,y),$$
$$\frac{dy}{dt} = g(x,y),$$

where f, g are continuous functions satisfying

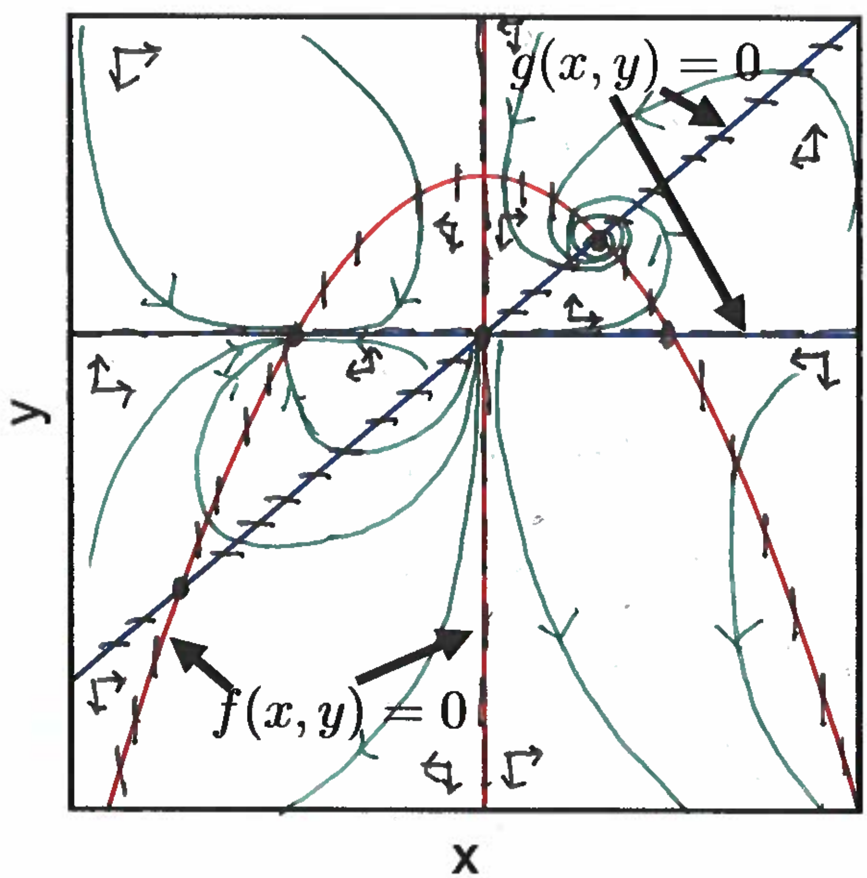
$$\lim_{x\to-\infty} f(x,y) = \infty,$$
$$\lim_{y\to\infty} g(x,y) = -\infty.$$

The figure below is a plot of the curves satisfying $f(x,y) = 0$ and $g(x,y) = 0$.

(a) (5 points) **Short Answer:** On this figure indicate any fixed points.

(b) (5 points) **Short Answer:** On this figure indicate the direction of the flow in the regions bounded by the curves $f(x,y) = 0$ and $g(x,y) = 0$.

(c) (5 points) **Short Answer:** Sketch a phase portrait on top of this figure. You should include enough solution trajectories so that all possible qualitatively different solution curves are represented.



5. (20 points) The following is a model for the spread of a sexually transmitted disease in a male and female population:

$$\begin{aligned}\dot{S}_F &= -\beta S_F I_M + \alpha I_F,\\ \dot{I}_F &= \beta S_F I_M - \alpha I_F,\\ \dot{S}_M &= -\beta S_M I_F + \alpha I_M,\\ \dot{I}_M &= \beta S_M I_F - \alpha I_M,\end{aligned}$$

where S_F, I_F, S_M, I_M denote the susceptible and infected populations of female and male populations and $\beta, \alpha > 0$ are parameters.

(a) (5 points) Show that $N_F = S_F + I_F$ and $N_M = S_M + I_F$ are constants in time.

$$\dot{N}_F = \dot{S}_F + \dot{I}_F = 0$$
$$\dot{N}_M = \dot{S}_M + \dot{I}_F = 0$$

(b) (5 points) Show that the above system of four differential equation can be reduced to the following system of differential equations:

$$\begin{aligned}\dot{I}_F &= \beta(N_F - I_F)I_M - \alpha I_F\\ \dot{I}_M &= \beta(N_M - I_M)I_F - \alpha I_M\end{aligned}$$

(c) (10 points) By computing the Jacobian for this system and calculating the eigenvalues, determine under what conditions the disease free state $I_F = I_M = 0$ is stable.

$$J = \begin{bmatrix} -\beta I_M - \alpha & \beta(N_F - I_F) \\ \beta(N_M - I_M) & -\beta I_f - \alpha \end{bmatrix}$$

$$\Rightarrow J(0,0) = \begin{bmatrix} -\alpha & \beta N_F \\ \beta N_M & -\alpha \end{bmatrix}$$

$$\lambda_{1,2} = \frac{-2\alpha \pm \sqrt{\alpha^2 - 4(\alpha^2 - \beta^2 N_F N_M)}}{2}$$

The fixed point is stable if

$$\alpha^2 - \beta^2 N_F M_M > 0$$
$$\Rightarrow \beta^2 N_F N_M / \alpha^2 < 1.$$

6. (20 points) The following is a proposed model for SIS dynamics with quarantining:

$$\dot{S} = -\beta IS - f(I,Q)S + \alpha Q + \alpha I,$$
$$\dot{I} = \beta IS - f(I,Q)I - \alpha I,$$
$$\dot{Q} = f(I,Q)(I+S) - \alpha Q,$$

where $\beta > 0$ and $\alpha > 0$ are constants and $f(I,Q)$ is a quarantining rate that depends on both the number of infected individuals and the number of quarantined individuals. This model has a constant population size of N.

(a) (5 points) **Short Answer:** If we first assume that f does not depend on Q, determine a functional form of $f(I)$ that satisfies $f(0) = 0$, $f(I) \geq 0$ on the domain $[0, N]$, and

$$\lim_{I\to\infty} f(I) = r,$$

where $r > 0$ is a constant. The function you select should be as simple as possible and be dimensionally consistent. No Dirac delta functions are needed.

$$f(I) = \frac{rI}{A+I}.$$

(b) (2.5 points) **Short Answer:** What does the assumption $\lim_{I\to\infty} f(I) = r$ in part (a) tell you about the quarantining rate in practical terms?

The quaranteening rate saturates as the number of infections grows due to limited resources.

(c) (5 points) **Short Answer** If we now assume that f depends on Q alone. Determine a functional form of $f(Q)$ that satisfies $f(0) = k$, k is a maximum of f on the domain $[0, N]$, $f(Q) \geq 0$ on the domain $[0, N]$, and $f(N) = 0$. Again, the function should be as simple as possible and be dimensionally consistent.

$$f(Q) = k(1 - Q/N)$$

(d) (2.5 points) **Short Answer:** What do the assumptions $f(0) = k$ and $f(N) = 0$ in part (c) tell you about the quarantining rate in practical terms?

The quarantining rate goes to zero as spaces fill up.

(e) (5 points) **Short Answer:** Using parts (a) and (c) construct a function $f(I,Q)$ on the domain $[0, N] \times [0, N]$ that satisfies $f(0,Q) = 0$, $f(I,N) = 0$, $f(I,Q) \geq 0$, and has a maximum at $f(N, 0)$.

$$f(I,Q) = \frac{rkI}{A+I}(1 - Q/N).$$