MST 383/683
Homework #1

Due Date: September 03, 2021

1. A communicable disease from which infectives do not recover may be modeled by the pair of differential equations

$$\dot{S} = -\beta SI,$$
$$\dot{I} = \beta SI.$$

Show that in a population of fixed size N, such a disease will eventually spread to the entire population, i.e. $\lim_{t\to\infty} I(t) = N$.

2. If a fraction λ of the population susceptible to a disease that provides immunity against reinfection moves out of the region of an epidemic, the situation may be modeled by a system

$$\dot{S} = -\beta SI - \lambda S,$$
$$\dot{I} = \beta SI - \alpha I.$$

Show that both S and I approach zero as $t \to \infty$.

3. Consider a disease spread by carriers who transmit the disease without exhibiting symptoms themselves. Let $C(t)$ be the number of carriers and suppose that the carriers are identified and isolated from contact with others at a constant per captia rate α, so that $\dot{C} = -\alpha C$. The rate at which susceptibles become infected is proportional to the number of carriers and to the number of susceptibles, so that $\dot{S} = -\beta SC$. Let C_0 and S_0 be the numbers of carriers and susceptibles, respectively, at time $t = 0$.

   (a) Determine the number of carriers at time t from the equation for C.

   (b) Substitute the solution to part (a) into the equation for S and determine the number of susceptibles at time t.

   (c) Find $\lim_{t\to\infty} S(t)$, the number of members of the population who escape the disease.

4. Consider the SIR model with births and deaths and with vaccination in place of recovery:

$$\dot{S} = \mu N - \frac{\beta}{N} SI - (\mu + \phi)S,$$
$$\dot{I} = \frac{\beta}{N} SI - (\mu + \gamma)I,$$
$$\dot{V} = \gamma I + \phi S - \mu V,$$

where N is the populations size, $\mu, \beta, \gamma > 0$, and $\phi \geq 0$.

   (a) Explain in practical terms what the constants μ, β, γ, and ϕ represent physically.

   (b) Show that $\frac{dN}{dt}$ =0. What does this result imply?

   (c) Discuss why it is enough to study the first two equations.

   (d) Determine under what conditions the number of infected individuals is growing in time at time $t = 0$ in the limit $S_0 \to N$. That is, determine under what conditions $\dot{I}(0) > 0$ in the limit $S_0 \to N$. Use this information to determine a quantity $R_0(\phi)$ such that if $R_0(\phi) > 1$ the number of infections is initially growing in time in the limit $S_0 \to N$.

   (e) Plot $R_0(\phi)$ and interpret your results in practical terms.

Homework #1

#1. A communicable disease from which infectives do not recover may be modeled by

$$\dot{S} = -\beta S I,$$
$$\dot{I} = \beta S I.$$

Show that in a population of fixed size N, $\lim_{t\to\infty} I(t) = N$.

Solution:

Since $\dot{S} + \dot{I} = 0$ it follows that $N = S + I = S_0 + I_0$ is constant in time. Consequently,

$$\dot{I} = \beta(N - I)I. \quad (*)$$

Therefore, $\dot{I}(0) = \beta S_0 I_0 > 0$ and thus I is monotone increasing at $t=0$. Moreover, by $(*)$ it follows that I is monotone increasing for all $0 < I < N$. Since $I_1^* = N$ is a solution it follows from existence and uniqueness that $I(t)$ is bounded above by N. Therefore, by the monotone convergence theorem there exists I_2^* such that $\lim_{t\to\infty} I(t) = I_2^*$. However, since $\lim_{t\to\infty} I(t) = I_2^*$ and $I(t)$ is monotone increasing it follows that $\lim_{t\to\infty} \dot{I}(t) = 0$. Therefore,

$$0 = \lim_{t\to\infty} \dot{I}(t) = \lim_{t\to\infty} \beta(N - I(t))I(t)$$
$$= \beta(N - I_2^*)I_2^*.$$

Therefore,

$$\lim_{t\to\infty} I(t) = I_2^* = N.$$

$\blacksquare$

#2.

If a fraction λ of the population susceptible to a disease that provides reinfection moves out of the region of an epidemic, the situation may be modeled by

$$\dot{S} = -\beta S I - \lambda S,$$
$$\dot{I} = \beta S I - \alpha I.$$

Show that both S and I approach zero as $t \to \infty$.

Solution:

Since $S^* = 0$ is a solution and $\dot{S} > 0$ if $S > 0$ it follows from similar arguments in #1 that $\lim_{t\to\infty} S(t) = 0$. Now,

$$\dot{I} = I(\beta S - \alpha).$$

Consequently, since $\lim_{t\to\infty} S(t) = 0$ it follows there exists t_1^* such that $t > t_1^*$ implies $S(t) < \alpha/\beta$ and thus I is monotone decreasing for $t > t_1^*$. Again $I^* = 0$ is a solution and thus from similar arguments in #1 it follows that

$$\lim_{t\to\infty} I(t) = 0.$$

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#3.

Consider a disease spread by carriers who transmit the disease without exhibiting symptoms themselves. The model is given by:

$$\dot{C} = -\alpha C$$
$$\dot{S} = -\beta S C.$$

(a) Determine the number of carriers at time t.

(b). Determine the number of susceptibles at time t.

(c) Find $\lim_{t\to\infty} S(t)$.

Solution:

(a). This is a linear equation with solution

$$C(t) = C_0 e^{-\alpha t}.$$

(b). Therefore,

$$\dot{S} = -\beta S C_0 e^{-\alpha t}$$

$$\Rightarrow \int_{S_0}^{S} -\frac{1}{\beta S}\,dS = \int_0^t C_0 e^{-\alpha t}\,dt$$

$$\Rightarrow -\frac{1}{\beta}\ln\left(\frac{S}{S_0}\right) = \frac{C_0}{-\alpha}\left(e^{-\alpha t}-1\right)$$

$$\Rightarrow \ln\left(\frac{S}{S_0}\right) = \frac{C_0\beta}{\alpha}\left(e^{-\alpha t}-1\right)$$

$$\Rightarrow S(t) = S_0 \exp\left(\frac{C_0\beta}{\alpha}\left(e^{-\alpha t}-1\right)\right).$$

(c). Computing it follows that:

$$\lim_{t\to\infty} S(t) = S_0 \exp\left(-\frac{C_0\beta}{\alpha}\right).$$

$\blacksquare$

#4

Consider the SIR model with births and deaths and with vaccination in place of recovery:

$$\dot{S} = \nu N - \frac{\beta}{N} SI - (\nu + \phi) S$$
$$\dot{I} = \frac{\beta}{N} SI - (\nu + \gamma) I$$
$$\dot{V} = \gamma I + \phi S - \nu V$$

Where N is the population size.

(a) Explain in practical terms what the constants ν, β, γ, ϕ represent physically.

(b) Show that $\frac{dN}{dt} = 0$. What does this result imply.

(c) Discuss why it is enough to study the first two equations.

(d). Calculate R_0 for this problem.

Solution:

(a). ν measures the birth rate. Note, this model assumes that the death rate is also ν. β is the rate of infection while γ, ϕ represent vaccination rates of the infected and susceptible populations respectively.

(b). $\dot{S} + \dot{I} + \dot{V} = \nu N - \nu S - \nu I - \nu V = \nu N - \nu N = 0.$

(c). Since V is decoupled from S and I it follows that S and I can be solved independently of $\dot{V}$.

(d) In the limit $S_0 \to N$ it follows that

$$\dot{I}(0) = I(0)(\beta - \nu - \gamma)$$

and thus the condition for an epidemic is

$$\beta - \nu - \gamma > 0$$

$$\Rightarrow \boxed{R_0 = \frac{\beta}{\nu + \gamma} > 1.}$$