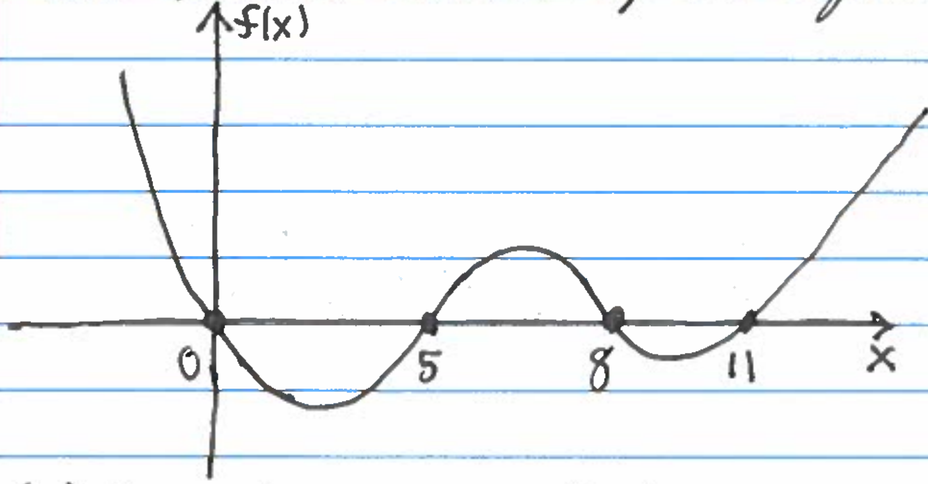
# Homework #2

## #1

A first-order differential equation is given by

$$\dot{x} = f(x),$$

where $f(x)$ is defined by the figure below.




(a) Determine the equilibria.
(b). Determine local stability.
(c). Graph the solution curves $x(t)$.
(d). What is the limit

$$\lim_{t\to\infty} x(t)$$

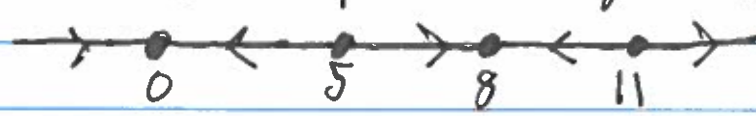
if $x(0)=15$? What about if $x(0)=1$?

**Solution:**

(a). The equilibria are given by

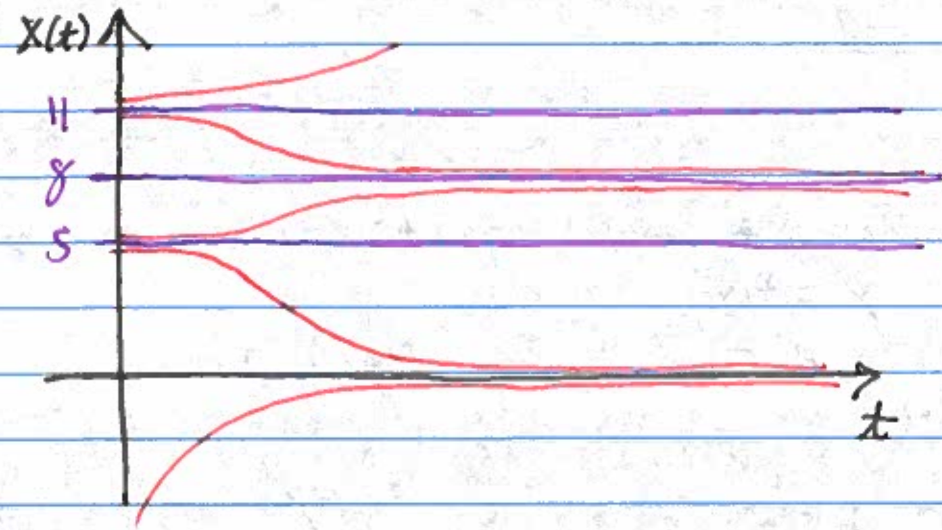
$$x^* = 0, 5, 8, 11.$$

(b) The phase portrait is given by:




Therefore, 0 and 8 are stable while 5 and 11 are unstable.

c.)

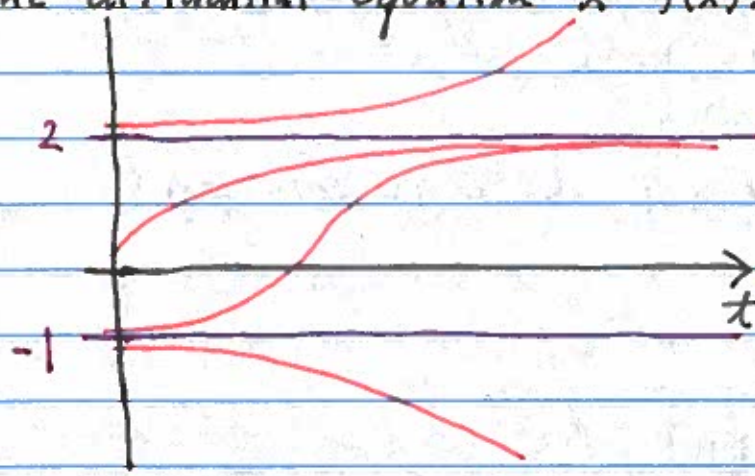



(d). If $x(0)=15$ it follows that $\lim_{t\to\infty} x(t)=\infty$.

If $x(0)=1$ it follows that $\lim_{t\to\infty} x(t)=0$.

#2

The curves $x(t)$ illustrated below correspond to solution curves for the differential equation $\dot{x}=f(x)$.



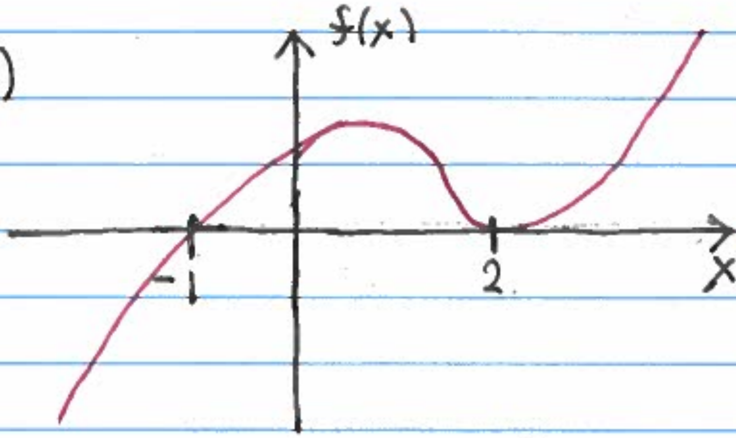

(a) Sketch a one dimensional phase portrait consistent with this figure.

(b) Sketch a graph of $f(x)$ consistent with this figure.

(c) Give a formula for $f(x)$ that is consistent with this figure.

Solution:

(a) 


(b) 


(c) $f(x) = (x+1)(x-2)^2$.

#3.

Consider the following SIS model:

$$\dot{S} = -\frac{\beta I^p S}{1+(\sigma I)^q} + \alpha I$$

$$\dot{I} = \frac{\beta I^p S}{1+(\sigma I)^q} - \alpha I$$

(a) Determine the dimensions of the constants.

(b) Reduce the SIS model to a single differential equation.

(c) Introduce appropriate dimensionless equations.

(d) For the case $p<1$, $q=p-1$, determine the threshold condition for the existence of endemic equilibria.

(e) For the case $p \geq 1$, $q=p$, determine the threshold condition for the existence of endemic equilibria.

(f) Why does this model not make sense if $p \geq 1$, $p \geq q$.

Solution:

(a) To balance dimensions it follows that

$$[\alpha] = \frac{1}{\text{time}}, \quad [\beta] = \frac{1}{\text{time} \cdot \text{pop}^p}, \quad [\sigma] = \frac{1}{\text{pop}}.$$

(b). Since $\dot{S} + \dot{I} = 0$, it follows from conservation of population that:

$$\dot{I} = \frac{\beta I^p (N - I)}{1 + (\sigma I)^q} - \alpha I.$$

(c) Letting $x = I/N$, $\tau = \alpha t$ it follows that:

$$N\alpha \frac{dx}{d\tau} = \frac{\beta N \cdot N^p (1-x) x^p}{1 + (\sigma N x)^q} - \alpha N x$$

$$\Rightarrow \frac{dx}{d\tau} = \frac{\beta N^p/\alpha \, (1-x) x^p}{1 + (\sigma N)^q x^q} - x.$$

$$\Rightarrow \frac{dx}{d\tau} = \frac{A(1-x)x^p}{1 + Bx^q} - x,$$

where $A = \beta N^p/\alpha$, $B = (\sigma N)^q$.

(d) Solving for the fixed points it follows that:

$$\frac{A(1-x)x^p}{1 + Bx^{p-1}} - x = 0$$

$$\Rightarrow A(1-x)x^p - x(1 + Bx^{p-1}) = 0 \quad (*)$$

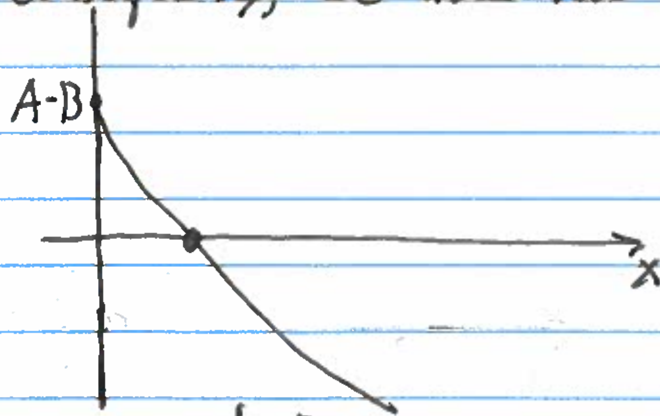
$$\Rightarrow x^p\left(A(1-x) - x^{1-p}(1 + Bx^{p-1})\right) = 0.$$

$$\Rightarrow x^p(A - Ax - x^{1-p} - B) = 0.$$

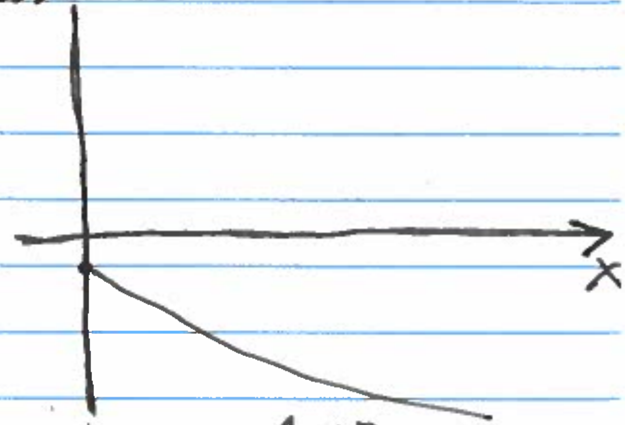
Letting $g(x) = A - Ax - x^{1-p} - B$ it follows that:

- $g(0) = A - B$.
- $\lim_{x \to \infty} g(x) = -\infty$.
- $g'(x) = -A - (1-p)x^{-p} < 0$.

Consequently, we have two cases:




$A>B$




$A<B$

Therefore, the threshold condition for an endemic equilibrium is that

$$A > B$$
$$\Rightarrow \frac{\beta N^p}{\alpha} > \sigma^{p-1} N^{p-1}$$
$$\Rightarrow \boxed{\frac{\beta N}{\alpha\,\sigma^{p-1}} > 1}$$

(e) Starting from the fixed points we have:

$$\frac{A(1-x)x^p}{1+Bx^p} - x = 0$$
$$\Rightarrow A(1-x)x^p - x(1+Bx^p) = 0$$
$$\Rightarrow x(A(1-x)x^{p-1} - 1 - Bx^p) = 0$$
$$\Rightarrow x(Ax^{p-1} - (A+B)x^p - 1) = 0$$

Letting $g(x) = Ax^{p-1} - (A+B)x^p - 1$ it follows that

- $g(0) = -1$
- $\lim_{x\to\infty} g(x) = -\infty$
- $g'(x) = (p-1)Ax^{p-2} - p(A+B)x^{p-1} = 0$
  $= x^{p-2}((p-1)A - p(A+B)x) = 0$

$g(x)$ has a critical point at

$$x^* = \frac{p-1}{p}\frac{A}{A+B}.$$

Furthermore,

$$g(x^*) = A\cdot\left[\left(\frac{p-1}{p}\right)\frac{A}{A+B}\right]^{p-1} - (A+B)\left[\left(\frac{p-1}{p}\right)\frac{A}{A+B}\right]^{p} - 1$$

$$= A\left(\frac{p-1}{p}\right)^{p-1}\frac{A^{p-1}}{(A+B)^{p-1}} - \left(\frac{p-1}{p}\right)^{p}\frac{A^p}{(A+B)^{p-1}} - 1$$

$$= \frac{A^p(p-1)^{p-1}\cdot p - (p-1)^pA^p - p^p(A+B)^{p-1})}{p^p(A+B)^{p-1}}$$

Consequently, $g(x^*)>0$ implies

$$A^p(p-1)^{p-1}p - (p-1)^pA^p - p^p(A+B)^{p-1} > 0$$

is the threshold condition.

$$\Rightarrow A^p(p-1)^{p-1}p > (p-1)^pA^p + p^p(A+B)^{p-1}.$$

(f) If $p>1$ and $p>q$ it follows that

$$\lim_{I\to\infty}\frac{\beta I^p}{1+(\sigma I)^q} = \infty,$$

that is the infection rate grows with infections which is contrary to the fact that with more infections there are less susceptibles to infect.

■

#4.

In cases of constant recruitment, the limiting system and the original system usually have the same qualitative dynamics. Consider the following SIV model with constant recruitment and vaccination:

$$\dot{S} = \Delta - \beta S \frac{I}{N} - \nu S$$

$$\dot{I} = \beta S \frac{I}{N} - (\nu + \gamma) I$$

$$\dot{V} = \gamma I - \nu V.$$

(a) What are the units of $\Delta$, $\beta$, $\nu$, and $\gamma$

(b) Interpret in practical terms what the constants represent physically.

(c) Find an equation for $\dot{N}$ and solve this equation.

(d) Show that $\lim N(t) = \Delta/\nu$

Solution:

(a) $[\Delta] = \text{pop}/\text{time}$

$[\beta] = 1/\text{time}$

$[\nu] = 1/\text{time}$

$[\gamma] = 1/\text{time}.$

(b) $\Delta$ is a constant per capita birth rate, $\beta$ is an infection rate, $\nu$ is a death rate, and $\gamma$ is a recovery rate.

(c). $\dot{N} = \dot{S} + \dot{I} + \dot{V} = \Delta - \nu N$. If we consider the homogeneous equation:

$$\dot{N}_H + \nu N_H = 0$$

it follows that the solution to the homogeneous equation is:

$$N_H = c\exp(-\nu t).$$

A particular solution to the homogeneous equation satisfies

$$\dot{N}_p + \nu N_p = \Delta$$

$$\Rightarrow N_p = \Delta/\nu.$$

Therefore, the general solution is given by:

$$N(t) = \Delta/\nu + c\exp(-\nu t).$$

(d). Clearly,

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \left(\Delta/\nu + c\exp(-\nu t)\right) = \Delta/\nu.$$

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#5.

A disease is introduced by two visitors into a town with 1200 inhabitants. An average infective is in contact with .4 inhabitants per day. The average duration of the infective period is 6 days, and recovered infectives are immune against reinfection. Assuming an SIR model, estimate how many inhabitants would have to be immunized to avoid an epidemic.

Solution:

Assuming a completely susceptible population it follows that

$$\begin{aligned} R_0 &= \text{\# of secondary infections} \\ &= .4 \cdot 6 \\ &= 2.4. \end{aligned}$$

Moreover,

$$R_0 = 2.4 = \frac{\beta N}{\alpha} = \beta \cdot 1200 \cdot 6$$

$$\Rightarrow \beta = \frac{2.4}{1200\cdot 6} = \frac{.4}{1200} = \frac{4\times 10^{-1}}{12\times 10^{2}} = \frac{1}{3}\cdot 10^{-3}.$$

Consequently, if we want to reduce the reproduction number below one we need

$$\frac{\beta S_0}{\alpha} < 1$$

$$\Rightarrow S_0 < \alpha/\beta$$

$$\Rightarrow S_0 < 1/6 \cdot 3 \cdot 10^3$$

$$= 500$$

Therefore, 700 people need to be immunized to guarantee the infection does not spread.

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