Homework #3

#3.2

Consider the following SIS model:

$$\dot{S} = \Delta - \beta I S + \alpha I - \nu S$$
$$\dot{I} = \beta I S - (\alpha + \gamma + \nu) I$$

(a) Sketch the nullclines of the system and direction of the vector field along them. Sketch the phase portrait.

(b). Determine the reproduction number and equilibria of the system.

(c) Calculate the Jacobian of each equilibrium and determine stability.

(d) Use the Dulac criterion to rule out periodic solutions.

(e) Use Poincare-Bendixon theorem to show convergence to equilibrium.

Solution:

Lets first nondimensionalize the system:

$$x = \frac{\nu}{\Delta} S, \quad y = \frac{\nu}{\Delta} I, \quad \tau = \alpha t$$

$$\Rightarrow \frac{\Delta \alpha}{\nu} \frac{dx}{d\tau} = \Delta - \frac{\Delta^2}{\nu^2} \beta x y + \frac{\alpha \Delta}{\nu} y - \frac{\nu \Delta}{\nu} x$$

$$\frac{\Delta \alpha}{\nu} \frac{dy}{d\tau} = \beta x y - (\alpha + \gamma + \nu) \frac{\Delta}{\nu} y$$

$$\Rightarrow \frac{dx}{d\tau} = \varsigma - R_0 x y + y - \varsigma x,$$

$$\frac{dy}{d\tau} = R_0 x y - (1 + \delta + \varsigma) y,$$

Where

$\varsigma = \mu/\alpha, \ \delta = \gamma/\alpha, \ R_0 = \beta\Lambda/\mu\alpha.$

Therefore, the nullclines are given by:

$\varsigma - R_0 xy + y - \varsigma x = 0$

$R_0 xy - (1+\delta+\varsigma)y = 0$

- $\frac{dx}{d\tau} = 0$:

$y(1 - R_0 x) = \varsigma(x-1)$

$\Rightarrow y = \dfrac{\varsigma(x-1)}{1-R_0 x}.$

- $\frac{dy}{d\tau} = 0$:

$y = 0$ or $x = \dfrac{(1+\delta+\varsigma)}{R_0} > \dfrac{1}{R_0}.$

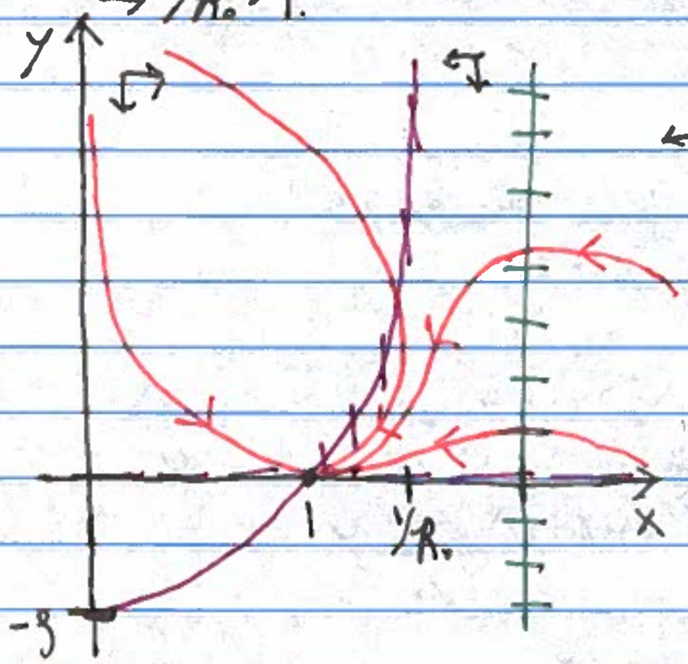
$\Rightarrow$ The fixed points are:

$(x_1^*, y_1^*) = (1, 0)$

$(x_2^*, y_2^*) = \left(\dfrac{1+\delta+\varsigma}{R_0}, \dfrac{\varsigma(R_0 - 1 - \delta - \varsigma)}{R_0(\delta+\varsigma)}\right)$

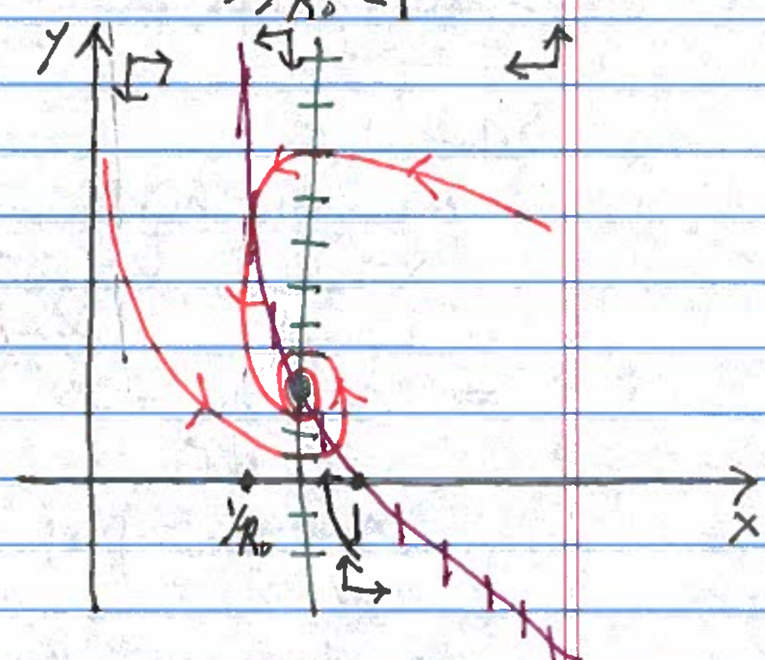
Case 1 ($R_0 < 1$):

$\Rightarrow 1/R_0 > 1.$

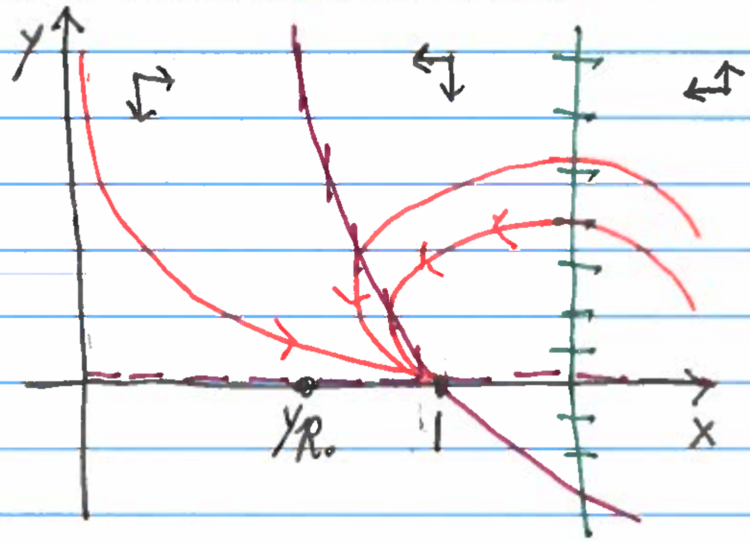


Case 2 ($R_0 > 1+\delta+\varsigma$)

$\Rightarrow 1/R_0 < 1$



Case 3 $(1 < R_0 < 1+\delta+s)$:



(b). In our units the disease free state $(1,0)$ is stable if $R_0 < 1+\delta+s$. Consequently, the basic reproduction number is

$$\overline{R}_0 = \frac{R_0}{1+\delta+s} = \frac{\Lambda\beta/\nu\alpha}{1+\gamma/\alpha+\nu/\alpha} = \frac{\Lambda\beta}{\nu(\alpha+\gamma+\nu)}.$$

(C) The Jacobian is given by:

$$J(x,y) = \begin{bmatrix} -R_0 y - s & -R_0 x + 1 \\ R_0 y & R_0 x - (1+\delta+s) \end{bmatrix}$$

$$\Rightarrow J(1,0) = \begin{bmatrix} -s & -R_0 + 1 \\ 0 & R_0 - (1+\delta+s) \end{bmatrix}$$

$$\Rightarrow \lambda_1, \lambda_2 = -s,\ R_0 - (1+\delta+s).$$

So $(1,0)$ is stable if $R_0 < (1+\delta+s)$.

(d) If we let

$$\vec{F}(x,y) = (s - R_0 xy + y - sx,\ R_0 xy - (1+\delta+s)y)$$

It follows that

$$\nabla\cdot\vec{F} = -R_0 y - s + R_0 x - (1+\delta+s)y.$$

However,

$$\nabla\cdot\left(\tfrac{1}{y}\vec{F}\right) = \tfrac{1}{y}(-R_0 y - s) < 0.$$

Therefore, by Dulac's criterion this system cannot have a limit cycle.

(e). For large values of y:

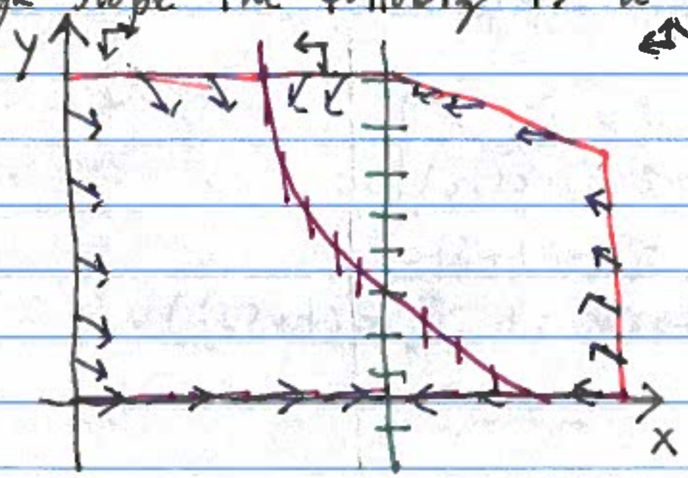
$$\frac{dy}{dx} \approx \frac{R_0 x - (1+\delta+s)}{1 - R_0 x}$$

$$= \frac{R_0 (x - (1+\delta+s)/R_0)}{1 - R_0 x}$$

If $x > \dfrac{1+\delta+s}{R_0} > \dfrac{1}{R_0}$ then

$$\frac{dy}{dx} < -c. \quad \left(\text{for large enough } y \text{ and } x > \frac{1+\delta+s}{R_0}\right)$$

Therefore, for sufficiently large values of y and steep enough slope the following is a trapping region.



#3.3.

Consider the following model:

$$\dot{S} = \Lambda - \beta(1+\nu I)IS - \nu S$$

$$\dot{I} = \beta(1+\nu I)IS - (\alpha+\nu)I$$

(a) Derive the reproduction number $\mathcal{R}_0$ for this model

(b) Show that $(\Lambda/\nu, 0)$ is locally stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.

(c) Argue that if $\mathcal{R}_0 > 1$, there is always a unique endemic equilibrium.

Solution:

(a) If we introduce a recovered class with dynamics

$$\dot{R} = \alpha I - \nu R$$

and let $N = S + I + R$ it follows that

$$\dot{N} = \Lambda - \nu N.$$

Therefore, the equilibrium population is

$$N = \Lambda/\nu.$$

Consequently,

$$\lim_{I\to 0,\, S\to \Lambda/\nu} \frac{\dot{I}}{I} = \beta\frac{\Lambda}{\nu} - (\alpha+\nu)$$

Therefore,

$$\mathcal{R}_0 = \frac{\beta\Lambda}{\nu(\alpha+\nu)}$$

(b) The Jacobian is given by:

$$J(S,I) = \begin{bmatrix} -\beta(1+\nu I)I - \nu & -\beta S - 2\nu IS\beta \\ \beta(1+\nu I)I & \beta S + 2\nu IS\beta - (\alpha+\nu) \end{bmatrix}$$

$$\rightarrow J(\Lambda/\nu, 0) = \begin{bmatrix} -\nu & -\beta\Lambda/\nu \\ 0 & \beta\Lambda/\nu - (\alpha+\nu) \end{bmatrix}$$

Therefore, the eigenvalues are

$$\lambda_1 = -\nu$$
$$\lambda_2 = \beta \Lambda/\nu - (\alpha + \nu)$$

Therefore, $(\Lambda/\nu, 0)$ is stable if $R_0 < 1$.

c.) Lets try solving for the fixed points:

$$0 = \Lambda - \beta(1+vI)IS - \nu S$$
$$0 = \beta(1+vI)IS - (\alpha+\nu)I.$$

The nullclines are:

N1:

$$S = \frac{\Lambda}{\nu + \beta(1+vI)I} = f_1(I)$$

N2:

$$I = 0 \text{ or } \beta(1+vI)S - (\alpha+\nu)$$
$$\Rightarrow S = \frac{\alpha+\nu}{\beta(1+vI)} = f_2(I).$$

Now,

$$f_1(0) = \frac{\Lambda}{\nu}, \lim_{I\to\infty} f_1(I) = 0$$

$$f_2(0) = \frac{\alpha+\nu}{\beta}, \lim_{I\to\infty} f_2(I) = 0.$$

However, $f_1(I)$ goes to 0 faster. Thus if

$$\frac{\Lambda}{\nu} > \frac{\alpha+\nu}{\beta}$$

then an endemic equilibrium exists, i.e., if

$$\frac{\Lambda\beta}{\nu(\alpha+\nu)} > 1$$

an endemic equilibrium exists.

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#3.4

Consider the following system:

$$\dot{S} = \Lambda - \frac{\beta I S}{1+\sigma S} + \alpha I - \nu S$$

$$\dot{I} = \frac{\beta I S}{1+\sigma S} - (\alpha+\nu) I$$

(a) What are the units of the parameters.

(b) Rescale the system into a nondimensional system.

(c) Determine conditions for the existence of an endemic equilibrium.

Solution:

(a) $[\Lambda] = \text{pop}/\text{time}$, $[\beta] = 1/\text{pop}\cdot\text{time}$, $[\alpha]=[\nu]=1/\text{time}$, $[\sigma] = 1/\text{pop}$.

(b) $x = \nu S/\Lambda$, $y = \nu I/\Lambda$, $\tau = \alpha t$.

$$\Rightarrow \frac{\alpha \Lambda}{\nu}\frac{dx}{d\tau} = \Lambda - \frac{\beta \Lambda^2 xy}{\nu^2(1+\frac{\sigma\Lambda}{\nu}x)} + \frac{\alpha\Lambda}{\nu} y - \frac{\nu\Lambda}{\nu} x$$

$$\frac{\alpha \Lambda}{\nu}\frac{dy}{d\tau} = \frac{\beta \Lambda^2}{\nu^2(1+\frac{\sigma\Lambda}{\nu}x)} - (\alpha+\nu)\frac{\Lambda}{\nu} y$$

$$\Rightarrow \frac{dx}{d\tau} = \xi - \frac{R_0 xy}{1+Ax} + y - \xi x,$$

$$\frac{dy}{d\tau} = \frac{R_0 xy}{1+Ax} - (1+\xi)y,$$

where

$$R_0 = \frac{\beta\Lambda}{\nu\alpha},\ A = \frac{\sigma\Lambda}{\nu},\ \xi = \frac{\nu}{\alpha}.$$

The nullclines are given by:

$$s - \frac{R_0 xy}{1+Ax} + y - sx = 0$$

$$\frac{R_0 xy}{1+Ax} - (1+s)y = 0$$

N1:

$$y\left(1 - \frac{R_0 x}{1+Ax}\right) = s(x-1)$$

$$\Rightarrow y = \frac{s(x-1)(1+Ax)}{1+(A-R_0)x}$$

$$= f(x).$$

N2:

$$y=0;\ \frac{R_0 x}{1+Ax} - (1+s) = 0$$

$$\Rightarrow y=0,\ R_0 x - (1+s)(1+Ax) = 0$$

$$(R_0 - (1+s)A)x = 1+s$$

$$\Rightarrow x = \frac{1+s}{R_0 - (1+s)A}$$

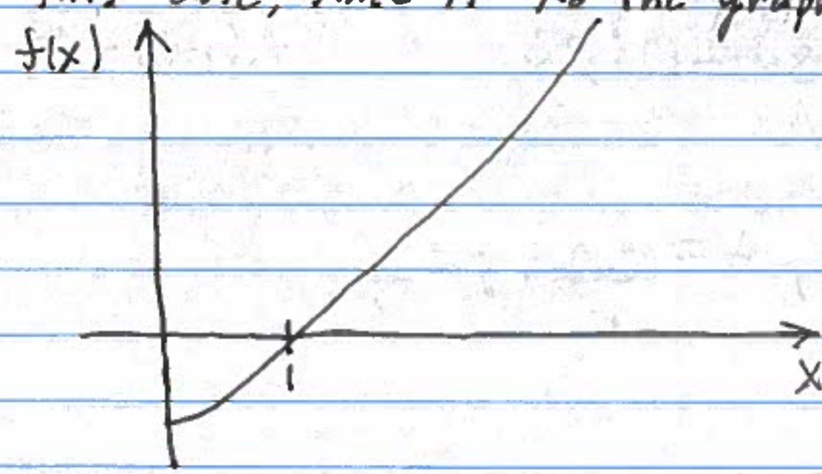
If we look at $f(x)$, it follows that

$$f(0) = -s,\ f(1) = 0,\ \lim_{x\to\infty} f(x) = \infty.$$

Therefore, in order for an endemic equilibrium to exist we must have that

$$R_0 - (1+s)A > 0 \Rightarrow A < \frac{R_0}{1+s} < R_0$$

In this case, since $A < R_0$ the graph of $f(x)$ is given by:



Consequently, an equilibrium will exist if

$$\frac{1+s}{R_0-(1+s)A} > 1$$

$$\Rightarrow (1+s) > R_0-(1+s)A$$

$$\Rightarrow (1+s)(1+A) > R_0$$

$$\Rightarrow \frac{(1+s)(1+A)}{R_0} > 1$$

Therefore, the condition for an endemic equilibrium to exist is

$$\frac{R_0}{A} > 1 \quad \text{and} \quad \frac{(1+s)(1+A)}{R_0} > 1.$$

#3.6

The following models for rabies

$$\dot{S} = rSe^{-aS} - \beta IS - \nu S$$

$$\dot{I} = \beta IS - (\alpha+\nu)I.$$

(a) Determine the reproduction number and equilibria of the system.

(b) Sketch the phase portrait.

(c). Calculate the Jacobian at each equilibrium and determine stability.

(d) Use the Dulac criterion to rule out periodic solutions

Solution:

(a) The nullclines are given by:

N1:

$$S=0, \; re^{-aS} - \beta I - \nu = 0$$

$$\Rightarrow S=0, \; I = \frac{re^{-aS}-\nu}{\beta}$$

N2:

$I=0,\ S=\dfrac{\alpha+\nu}{\beta}.$

Therefore, the fixed points are:

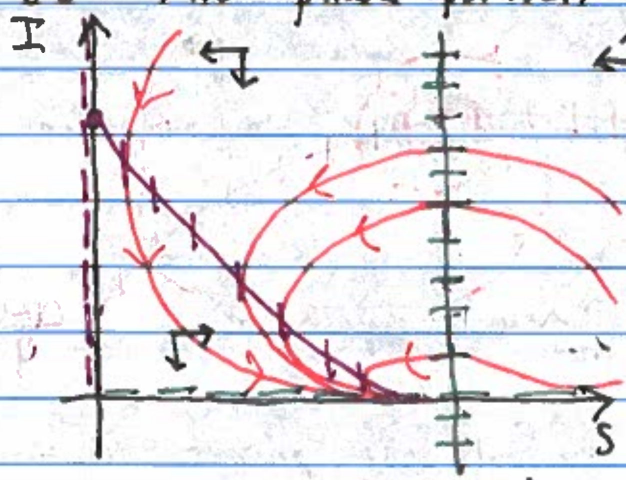
- $I=0,\ S=0$
- $I=0,\ S=-\frac{1}{a}\ln(\nu\beta/r)=\frac{1}{a}\ln(r/\nu\beta).$
- $S=(\alpha+\nu)/\beta,\ I=\dfrac{re^{-a(\alpha+\nu)/\beta}-\nu}{\beta}.$

Now,

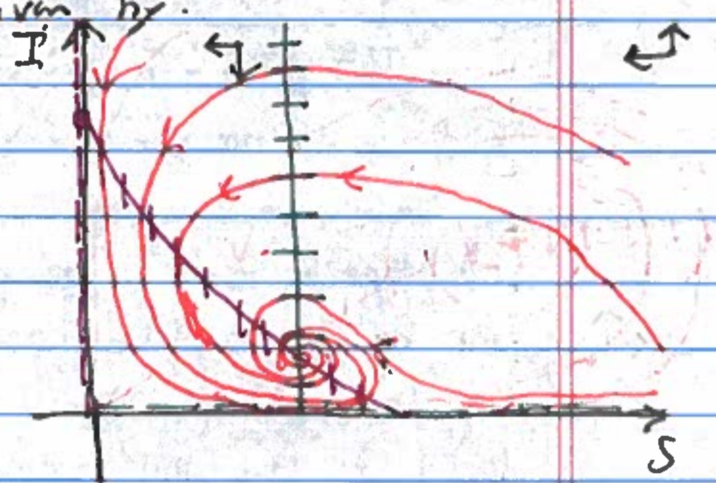
$$\lim_{I\to 0}\dot{I}/I=\beta S-(\alpha+\nu)$$

$$\Rightarrow R_0=\frac{\beta S_0}{\alpha+\nu}.$$

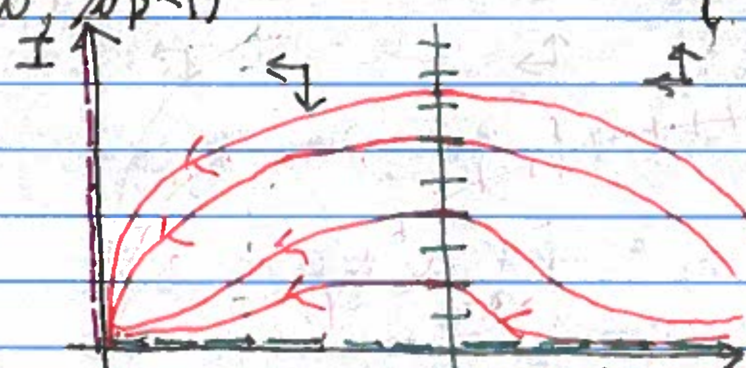
(b). The phase portrait is given by:



$(r>\nu,\ r/\nu\beta<1)$



$(r>\nu,\ r/\nu\beta>1)$



$(\nu<r,\ \alpha+\nu<\beta)$

(c) The Jacobian is given by

$$J(S,I) = \begin{bmatrix} re^{-aS} - raSe^{-aS} - \beta I - \nu & -\beta S \\ \beta I & \beta S - (\alpha+\nu) \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} r-\nu & 0 \\ 0 & -(\alpha+\nu) \end{bmatrix}$$

Therefore, $(0,0)$ is stable if $r<\nu$.

$$J\left(\tfrac{1}{a}\ln\left(\tfrac{r}{\nu\beta}\right), 0\right) = \begin{bmatrix} r\left(-\tfrac{r}{\nu\beta}\right) - r\ln\left(\tfrac{r}{\nu\beta}\right)\left(\tfrac{r}{\nu\beta}\right) - \nu & * \\ 0 & * \end{bmatrix}$$

$\vdots$

(d) $\nabla \cdot \vec{F} = re^{-aS} - raSe^{-aS} - \beta I - \nu + \beta - (\alpha+\omega).$

However,

$$\nabla \cdot \left(\frac{1}{IS}\vec{F}\right) = -\frac{are^{-aS}}{I} < 0.$$

By Dulac's criterion no limit cycle exists in the first quadrant.

$\blacksquare$