MST 383/683
Homework #4

Due Date: October 15 2021

1. Consider an SIS network model with adjacency matrix A and where each node has a status $X_i(t) \in \{0,1\}$ denoting whether node i is susceptible or infected at time t. Suppose further that the transition probabilities for the status of each node are given by:

$$P(X_i(t+\Delta t) = 1|X_i(t) = 0) = \beta \Delta t \sum_{j=1}^{n} A_{ij} X_j(t)$$
$$P(X_i(t+\Delta t) = 0|X_i(t) = 0) = 1 - \beta \Delta t \sum_{j=1}^{n} A_{ij} X_j(t)$$
$$P(X_i(t+\Delta t) = 0|X_i(t) = 1) = \alpha \Delta t$$
$$P(X_i(t+\Delta t) = 1|X_i(t) = 1) = 1 - \alpha \Delta t$$

   (a) Prove that $[SS](t) + [SI](t) = \langle k_S(t)\rangle [S](t)$ and $[SI] + [II] = \langle k_I(t)\rangle [I](t)$, where $\langle k_S(t)\rangle$ and $\langle k_I(t)\rangle$ denote the average degree of the susceptible and infected nodes.

   (b) Prove that $[SSI] + [ISI] = (\langle k_S(t)\rangle - 1)[SI]$.

2. Consider an SIR network model with adjacency matrix A and where each node has a status $X_i(t) \in \{0,1,2\}$ denoting whether node i is susceptible, infected, or recovered at time t. Suppose further that the transition probabilities for the status of each node are given by:

$$P(X_i(t+\Delta t) = 0|X_i(t) = 0) = 1 - \beta \Delta t \sum_{j=1}^{n} A_{ij} X_j(t) \mathbb{1}\{x_j = 1\}.$$
$$P(X_i(t+\Delta t) = 1|X_i(t) = 0) = \beta \Delta t \sum_{j=1}^{n} A_{ij} X_j(t) \mathbb{1}\{x_j = 1\}.$$
$$P(X_i(t+\Delta t) = 2|X_i(t) = 0) = 0$$
$$P(X_i(t+\Delta t) = 0|X_i(t) = 1) = 0$$
$$P(X_i(t+\Delta t) = 1|X_i(t) = 1) = 1 - \alpha \Delta t$$
$$P(X_i(t+\Delta t) = 2|X_i(t) = 1) = \alpha \Delta t$$
$$P(X_i(t+\Delta t) = 0|X_i(t) = 2) = 0$$
$$P(X_i(t+\Delta t) = 1|X_i(t) = 2) = 0$$
$$P(X_i(t+\Delta t) = 2|X_i(t) = 2) = 1$$

   Following the derivation we did for the SIS network model, show that

$$[\dot{S}] = -\beta [SI],$$
$$[\dot{I}] = \beta [SI] - \alpha [I],$$
$$[\dot{R}] = \alpha [I].$$

   Note, you just need to reproduce what we did in class in detail.

3. The differential equations for the dynamics of the nodes and edges for the SIS model are given by

$$\begin{aligned}
[\dot{S}] &= -\beta[SI] + \alpha[I] \\
[\dot{I}] &= \beta[SI] - \alpha[I] \\
[\dot{SS}] &= -2\beta[SSI] + 2\alpha[SI] \\
[\dot{SI}] &= \beta([SSI] - [ISI] - [SI]) \\
[\dot{II}] &= 2\beta([ISI] + [IS]) - 2\alpha[II]
\end{aligned}$$

(a) Using the following approximation:

$$[ABC] \approx \frac{\langle k \rangle - 1}{\langle k \rangle} \frac{[AB][BC]}{[B]},$$

derive a closed system of equations for the dynamics of the nodes and edges.

(b) Show that $[S] + [I]$ and $[SS] + 2[SI] + [II]$ are conserved quantities and thus

$$\begin{aligned}
n &= [S](0) + [I](0), \\
n\langle k \rangle &= [SS](0) + 2[SI](0) + [II](0)
\end{aligned}$$

are constant in time.

(c) Using the result of part (b), reduce the system of equations derived in part (a) to a system of three differential equations for $[S], [SS]$, and $[SI]$.

(d) Assuming further that $[SI] = \langle k \rangle [S] - [SS]$, reduce this system further to a system of two differential equations.

(e) For the system of differential equations you have derived, determine the fixed points, analyze their stability, and sketch phase portraits that illustrate all of the qualitatively different cases that occur.

(f) Calculate a dimensionless parameter $\mathcal{R}_0$ such that if $\mathcal{R}_0 > 1$ the disease becomes endemic.

4. For the SIR model defined in problem #2, derive differential equations for the edges:

$$[SS], [SI], [SR], [II], [IR], [RR].$$

5. In this problem you will derive differential equations for an SIS model in which edges can be paused. Specifically, individuals will pause connections at a rate proportional to the total number of infections. This could model a disease in which infected individuals are asymptomatic but as more infections are reported individuals remove themselves from the network.

(a) For the SIS model given in problem #3, introduce a new class of edges $\overline{[SS]}$, $\overline{[IS]}$, and $\overline{[II]}$ which denote paused connections. Develop a system of eight differential equations for the dynamics of $[S], [I], [SS], [SI], [II], \overline{[SS]}, \overline{[SI]}$ and $\overline{[II]}$ in which paused connections are introduced at rate proportional to the number of infections. Note, your equations should be conservative in the sense that in addition to $[S] + [I]$, $[SS] + 2[SI] + [II] + \overline{[SS]} + 2\overline{[SI]} + \overline{[II]}$ is a conserved quantity.

(b) Using the same moment closure approximation defined in problem #3(a), close this system of equations.

(c) Using the conserved quantities reduce this system of equations to a set of six differential equations.

(d) Calculate a dimensionless parameter $\mathcal{R}_0$ such that if $\mathcal{R}_0 > 1$ the disease becomes endemic.

Homework #4

#1

Consider an SIS network model.

(a) Prove that $[SS]+[SI]=\langle K_S\rangle[S]$, and in addition $[SI]+[II]=\langle K_I\rangle[I]$.

(b) Prove that $[SSI]+[ISI]=(\langle K_S\rangle-1)[SI]$.

Solution:

(a) Note,

$$\langle K_S\rangle=\frac{1}{[S]}\sum_{i=1}^{n}\sum_{j=1}^{n}P(I_i=0)A_{ij}.$$

Therefore,

$$[SS]+[SI]=\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(P(I_i=0,I_j=0)+P(I_i=0,I_j=1))$$

$$=\sum_{i=1}A_{ij}\ P(I_i=0)$$

$$=[S]\langle K_s\rangle.$$

(b.) Note, $(\langle K_s\rangle-1)[SI]$ denotes the average number of connections to the S node of the SI edges. Consequently, if we let $\langle K^S_{SI}\rangle$ denote the average number of links to an S node which is part of an SI edge we know that

$$\langle K^S_{SI}\rangle=(\langle K_S\rangle-1)=\frac{1}{[SI]}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}A_{ij}A_{jk}P(I_j=0,I_k=1)$$

Therefore,

$$[SSI]+[ISI]=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}A_{ij}A_{jk}(P(I_i=0,I_j=0,I_k=1)$$
$$+P(I_i=1,I_j=0,I_k=1)$$

$$\Rightarrow [SSI]+[ISI] = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} A_{ij}A_{jk}(P(I_j=0, I_k=1)$$

$$= [SI](\langle k_S\rangle - 1).$$

■

#2.

Show that the dynamics for the nodes of an SIR network satisfy:

$$[\dot{S}] = -\beta[SI]$$
$$[\dot{I}] = \beta[SI] - \alpha[I]$$
$$[\dot{R}] = \alpha[I].$$

Solution:

- $$[S](t+\Delta t) = \sum_{i=1}^{n} P(I_i(t+\Delta t)=0)$$

$$= \sum_{i=1}^{n} P(I_i(t+\Delta t)=0 \mid I_i(t)=0)P(I_i(t)=0)$$

$$= \sum_{i=1}^{n}\left(1-\beta\Delta t\sum_{j=1}^{n} A_{ij}I_i(t)\right)P(I_i(t)=0).$$

$$\Rightarrow \frac{[S](t+\Delta t)-[S](t)}{\Delta t} = -\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}I_j(t)P(I_i(t)=0)$$

Taking $\Delta t \to 0$ we obtain:

$$[\dot{S}] = -\beta[SI].$$

- $$[I](t+\Delta t) = \sum_{i=1}^{n} P(I_i(t+\Delta t)=1)$$

$$= \sum_{i=1}^{n}[P(I_i(t+\Delta t)=1 \mid I_i(t)=0)P(I_i(t)=0)$$
$$+ P(I_i(t+\Delta t)=1 \mid I_i(t)=1)P(I_i(t)=1)$$

$$\Rightarrow [I](t+\Delta t) = \sum_{i=1}^{n} \beta \Delta t \sum_{j=1}^{n} A_{ij} I_j(t) P(I_i(t)=0) + \sum_{i=1}^{n} (1-\alpha\Delta t) P(I_i(t)=1)$$

$$\Rightarrow \frac{[I](t+\Delta t)-[I](t)}{\Delta t} = \sum_{i=1}^{n}\sum_{j=1}^{n} \beta A_{ij} I_j(t) P(I_i(t)=0) - \alpha \sum_{i=1}^{n} P(I_i(t)=1).$$

Taking $\Delta t \to 0$ we obtain:

$$[\dot{I}] = +\beta[SI] - \alpha[I].$$

$$-[R](t+\Delta t) = \sum_{i=1}^{n} P(I_i(t+\Delta t)=2)$$

$$= \sum_{i=1}^{n}\Big[P(I_i(t+\Delta t)=2 \mid I_i(t)=1) P(I_i(t)=1) + P(I_i(t+\Delta t)=2 \mid I_i(t)=2) P(I_i(t)=2)\Big]$$

$$= \sum_{i=1}^{n}[\alpha \Delta t P(I_i(t)=1) + P(I_i(t)=2)]$$

$$\Rightarrow \frac{[R](t+\Delta t)-[R](t)}{\Delta t} = \alpha \sum_{i=1}^{n} P(I_i(t)=1)$$

Taking the limit as $\Delta t \to 0$ we obtain:

$$[\dot{R}] = \alpha[I].$$

■

#3.

Analyze the dynamics of the SIS model with edge dynamics:

$$[\dot{S}] = -\beta[SI] + \alpha[I]$$
$$[\dot{I}] = \beta[SI] - \alpha[I]$$
$$[\dot{SS}] = -2\beta[SSI] + 2\alpha[SI]$$
$$[\dot{SI}] = \beta([SSI] - [ISI] - [SI]) + \alpha([II] - [SI])$$
$$[\dot{II}] = 2\beta([ISI] + [IS]) - 2\alpha[II]$$

Solution:

(a-b) By construction $n = [S] + [I]$ and $[SS] + 2[SI] + [II] = n\langle k\rangle$ are conserved quantities and thus $[I]$ and $[II]$ can be eliminated yielding:

$$[\dot{S}] = -\beta[SI] + \alpha(n - [S]).$$
$$[\dot{SS}] = -2\beta[SSI] + 2\alpha[SI]$$
$$[\dot{SI}] = \beta([SSI] - [ISI] - [SI]) + \alpha(n\langle k\rangle - 3[SI] - [SS]).$$

If we further assume that:

$$[SI] = \langle k\rangle[S] - [SS]$$

and apply the standard moment closure we have that:

$$[\dot{S}] = -\beta[SI] + \alpha(n - [S])$$
$$[\dot{SS}] = -2\beta\frac{(\langle k\rangle - 1)}{\langle k\rangle}\frac{[SS]\cdot[SI]}{[S]} + 2\alpha[SI]$$

$$\Rightarrow [\dot{S}] = -\beta(\langle k\rangle[S] - [SS]) + \alpha(n - [S])$$
$$[\dot{SS}] = -2\beta\frac{(\langle k\rangle - 1)}{\langle k\rangle}\frac{[SS](\langle k\rangle[S] - [SS])}{[S]} + 2\alpha(\langle k\rangle[S] - [SS])$$

If we make the change of variables

$$x = \frac{[S]}{n}, \quad y = \frac{[SS]}{n\langle k\rangle}.$$

It follows that

$$n\alpha \frac{dx}{d\tau} = -\beta \langle k \rangle n(x-y) + \alpha n(1-x)$$

$$n\alpha \langle k \rangle \frac{dz}{d\tau} = -2\beta \frac{(\langle k \rangle -1) n^2 \langle k \rangle^2 y (x-y)}{n \langle k \rangle x} + 2\alpha n \langle k \rangle (x-y)$$

$$\Rightarrow \frac{dx}{d\tau} = -\frac{\beta \langle k \rangle}{\alpha}(x-y) + (1-x)$$

$$\frac{dy}{d\tau} = -\frac{2\beta}{\alpha}(\langle k \rangle -1) \frac{y(x-y)}{x} + 2(x-y)$$

Letting $R_0 = \beta \langle k \rangle / \alpha$ and $\gamma = (\langle k \rangle - 1)/\langle k \rangle$, we obtain the system:

$$\boxed{\begin{aligned} \frac{dx}{d\tau} &= -R_0(x-y) + (1-x) \\ \frac{dy}{d\tau} &= -2R_0 \gamma \frac{y(x-y)}{x} + 2(x-y). \end{aligned}}$$

Note, in a system in which n is large and everybody is connected we obtain

$$R_0 \simeq \beta n / \alpha \quad \text{and} \quad \gamma \approx 1$$

as expected.

We now calculate the nullclines:

N1:

$$\frac{dx}{d\tau} = 0 \Rightarrow R_0 y = (R_0 + 1)x - 1$$

$$\Rightarrow y = (1 + 1/R_0)x - 1/R_0$$

N2:

$$\frac{dy}{d\tau}=0 \Rightarrow y=x \text{ and } 1-R_0\gamma\frac{y}{x}$$

$$\Rightarrow y=x \text{ and } y=\frac{1}{R_0\gamma}x.$$

To determine fixed points we have:

- $y=x$ and $y=(1+1/R_0)x-1/R_0$
  $$\Rightarrow x=1 \text{ and } y=1$$
- $y=\frac{1}{R_0\gamma}x$ and $y=(1+1/R_0)x-1/R_0$
  $$\Rightarrow \frac{1}{R_0\gamma}x=(1+1/R_0)x-1/R_0$$
  $$\Rightarrow x=(R_0\gamma+\gamma)x-\gamma$$
  $$\Rightarrow x=\frac{\gamma}{\gamma(1+R_0)-1} \text{ and } y=\frac{1}{R_0(\gamma(1+R_0)-1)}.$$

Thus, the second fixed point will exist if
$$\gamma(1+R_0)>1.$$

The Jacobian for this system is given by:

$$J(x,y)=\begin{bmatrix} -R_0-1 & R_0 \\ 2(1-\gamma R_0 y/x)+2(x-y)(\gamma R_0 y/x^2) & 4\gamma R_0 y/x-2\gamma R_0-2 \end{bmatrix}$$

$$\Rightarrow J(1,1)=\begin{bmatrix} -R_0-1 & R_0 \\ 2(1-\gamma R_0) & 2(\gamma R_0-1) \end{bmatrix}$$

Therefore, $(1,1)$ will be stable if:

$$-R_0 - 1 + 2(\gamma R_0 - 1) < 0 \text{ and } 2(R_0+1)(1-\gamma R_0) - 2R_0(1-\gamma R_0)x$$

$$\Rightarrow R_0(2\gamma - 1) - 3 < 0 \quad \text{and} \quad 2(1-\gamma R_0) > 0$$

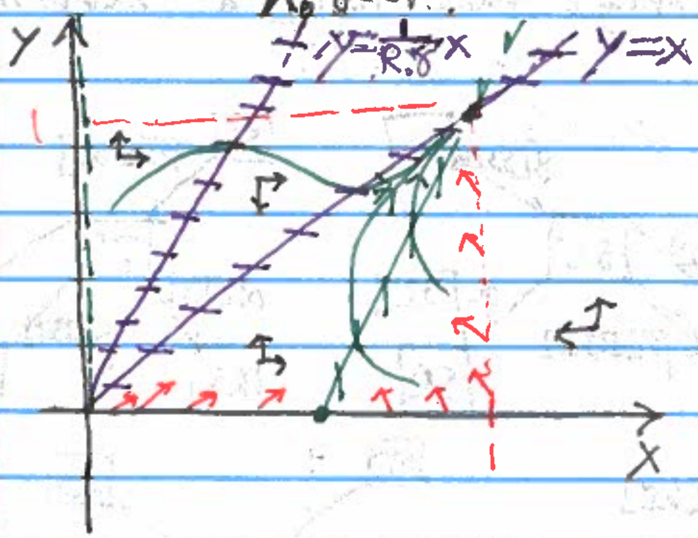
$$\Rightarrow R_0(2\gamma - 1) - 3 < 0 \quad \text{and} \quad \gamma R_0 < 1$$

Now, if $\gamma R_0 > 1$ it follows that

$$R_0(2\gamma - 1) - 3 > 2 - 3 - R_0 = -1 - R_0 < 0$$

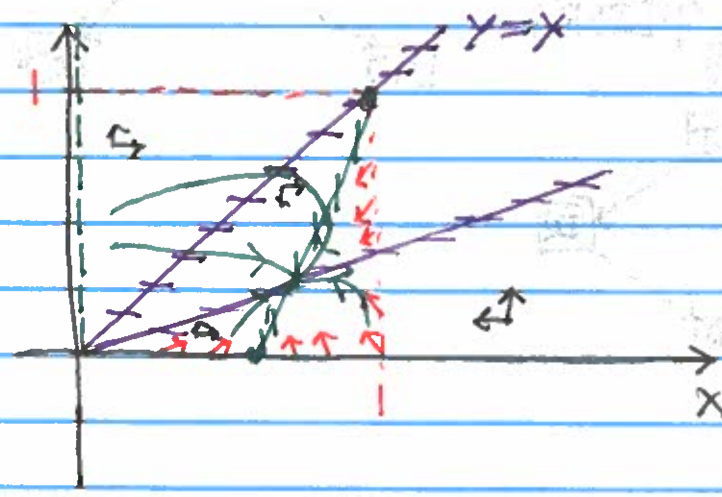
Consequently, the condition for stability of the disease free equilibrium is

$$R_0 \gamma < 1.$$



(Disease free Equilibrium)

Case 1: $R_0 \gamma < 1$



(Endemic Equilibrium)

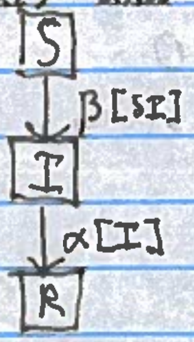
Case 2: $R_0 \gamma > 1$

#4

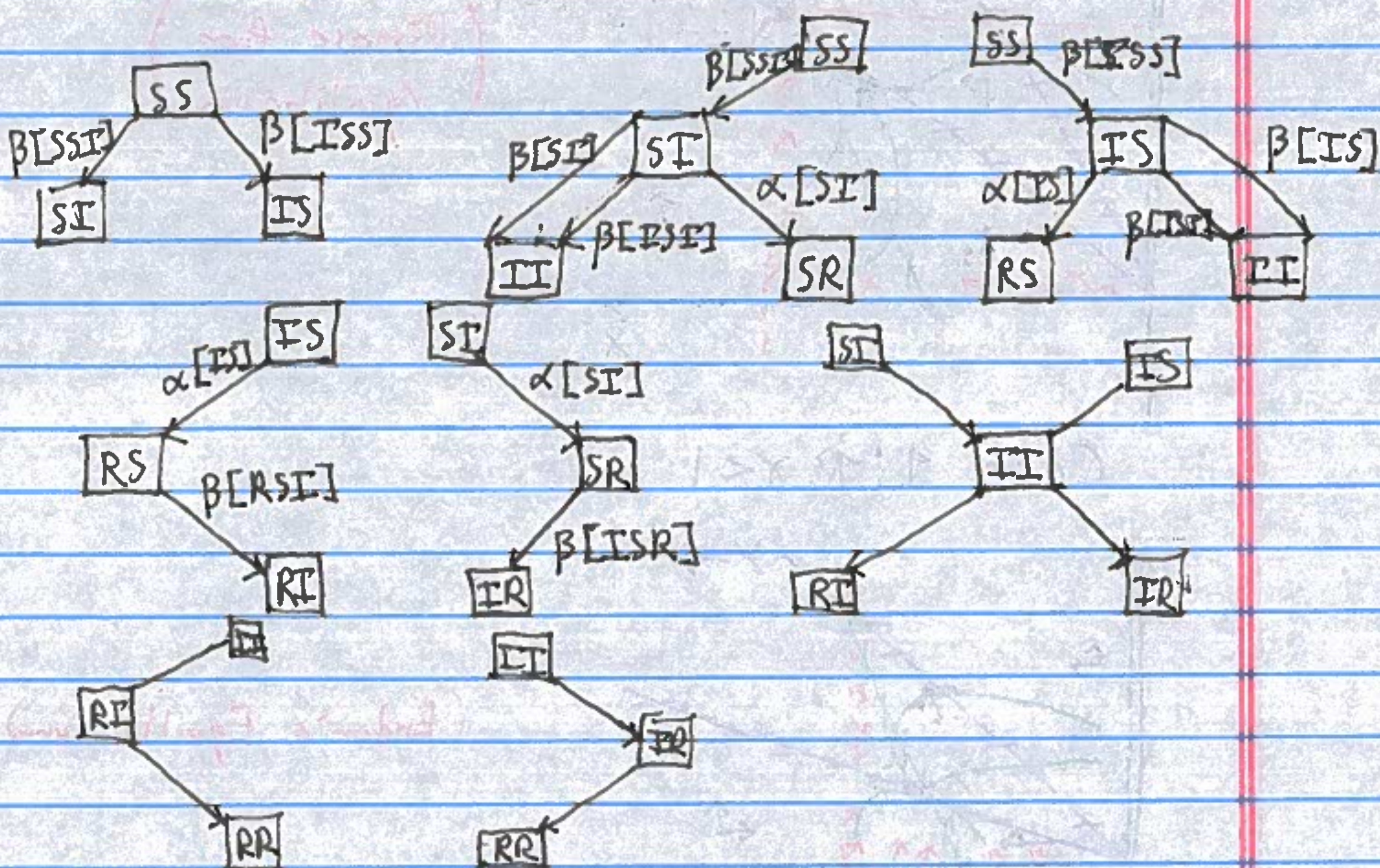
Derive a model for the nodes and edges in an SIR model.

Solution:

We first draw a diagram to illustrate the flows between nodes and edges



To build the model for the edges I am going to look at the inflows and outflows of each compartment and then try to arrange them into a network.

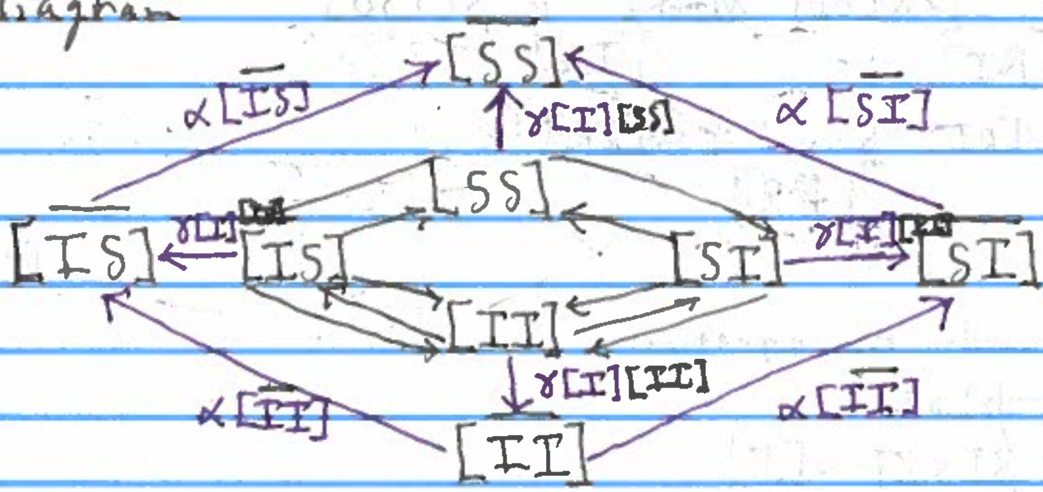


#5

In this problem we derive and analyze an SIS model with paused connections.

Solution:

If we assume paused connections are introduced at a rate proportional to infections we have the following diagram



$$\Rightarrow \dot{[S]} = -\beta[SI] + \alpha[I]$$
$$\dot{[I]} = \beta[SI] - \alpha[I]$$
$$\dot{[SS]} = -2\beta[SSI] + 2\alpha[SI] - \gamma[I][SS]$$
$$\dot{[SI]} = \beta[SSI] - \beta[ISI] - \beta[SI] + \alpha[II] - \gamma[I][SI]$$
$$\dot{[II]} = 2\beta[SSI] + 2\beta[SI] - 2\alpha[II] - \gamma[I][II]$$

$$\dot{\overline{[SS]}} = \gamma[I]\cdot[SS] + 2\alpha\overline{[SI]}$$
$$\dot{\overline{[SI]}} = \gamma[I][SI] + \alpha[II] - \alpha[SI]$$
$$\dot{\overline{[II]}} = \gamma[I][II] - 2\alpha\overline{[II]}$$

(This is a bonus problem as I made a mistake)

This motivates the construction of the following diagram

$$
\begin{tikzcd}
 & & & {[SS]} \arrow[dl, "\beta[ISS]"'] \arrow[dr, "\beta[SSI]"] & & \\
 & & {[IS]} \arrow[dl, "\alpha[IS]"'] \arrow[dr, shift left, "\beta[IS]"] \arrow[dr, shift right, "\beta[ISI]"'] & & {[SI]} \arrow[dl, shift right, "\beta[SI]"'] \arrow[dl, shift left, "\beta[ISI]"] \arrow[dr, "\alpha[SI]"] & \\
 & {[RS]} \arrow[dr, "\beta[RSI]"'] & & {[II]} \arrow[dl, "\alpha[II]"] \arrow[dr, "\alpha[II]"'] & & {[SR]} \arrow[dl, "\beta[ISR]"] \\
 & & {[RI]} \arrow[dr, "\alpha[RI]"'] & & {[IR]} \arrow[dl, "\alpha[IR]"] & \\
 & & & {[RR]} & &
\end{tikzcd}
$$

This yields the equations

$$[\dot{S}] = -\beta[SI]$$
$$[\dot{I}] = \beta[SI] - \alpha[I]$$
$$[\dot{R}] = \alpha[I]$$

$$[\dot{SS}] = -2\beta[SSI]$$
$$[\dot{SI}] = -\beta[SI] - \beta[ISI] - \alpha[SI]$$
$$[\dot{II}] = 2\beta[SI] + 2\beta[ISI] - 2\alpha[II]$$
$$[\dot{SR}] = \alpha[SI] - \beta[ISR]$$
$$[\dot{IR}] = \beta[ISR] + \alpha[II] - \alpha[IR]$$
$$[\dot{RR}] = 2\alpha[IR]$$

$\blacksquare$