Homework #6

#1.

$$\min_{u} \int_1^2 \left((t u(t))^2 + t^2 x(t)^2 \right) dt$$

$$\dot{x} = -u(t)$$
$$x(1) = 1.$$

Solution:

(a). Letting
$$C[u] = \int_1^2 \left((t u(t))^2 + t^2 x(t)^2 \right) dt$$
and for a perturbation $\tilde{u}$ define
$$\begin{aligned}
f(\varepsilon) &= C[u + \varepsilon \tilde{u}] \\
&= \int_1^2 \left((t(u(t) + \varepsilon \tilde{u}))^2 + t^2 x(t,\varepsilon)^2 \right) dt \\
&\quad + \int_1^2 \frac{d}{dt}\left(\lambda \cdot x(t,\varepsilon) \right) dt - \lambda(2) x(2,\varepsilon) + \lambda(1) x(1,\varepsilon) \\
&= \int_1^2 \left(t(u(t) + \varepsilon \tilde{u})^2 + t^2 x(t,\varepsilon)^2 \right) dt \\
&\quad + \int_1^2 \left[\lambda' \cdot x(t,\varepsilon) + \lambda \dot{x}(t,\varepsilon) \right] dt - \lambda(2) x(2,\varepsilon) + \lambda(1) \\
&= \int_1^2 \left(t(u(t) + \varepsilon \tilde{u})^2 + t^2 x(t,\varepsilon)^2 \right) dt \\
&\quad + \int_1^2 \left[\lambda' x(t,\varepsilon) - \lambda (u + \varepsilon \tilde{u}) \right] dt - \lambda(2) x(2,\varepsilon) + \lambda(1).
\end{aligned}$$

$$\Rightarrow f'(\varepsilon) = \int_1^2 \left[2t(u(t) + \varepsilon \tilde{u}) \tilde{u} + 2t^2 x \frac{\partial x}{\partial \varepsilon} + \lambda' \frac{\partial x}{\partial \varepsilon} - \lambda \tilde{u} \right] dt - \lambda(2) \left. \frac{\partial x}{\partial \varepsilon} \right|_{x=2}$$

$$\Rightarrow f'(0) = \int_1^2 \left[(2tu - \lambda) \tilde{u} + (2t^2 x + \lambda') \frac{\partial x}{\partial \varepsilon} \right] dt - \lambda(2) \frac{\partial x}{\partial \varepsilon}$$

Therefore, we obtain the additional conditions
$$u = \lambda / 2t, \quad \lambda(2) = 0,$$
$$\dot{\lambda} = -2t^2 x.$$

b.) The Hamiltonian for this system is given by:

$$H(x,u,\lambda) = t u^2 + t^2 x^2 - \lambda u$$

$$\Rightarrow \quad \dot{x} = \frac{\partial H}{\partial \lambda} = -u$$

$$\Rightarrow \quad \dot{\lambda} = -\frac{\partial H}{\partial x} = -2tx$$

$$\Rightarrow \quad 0 = \frac{\partial H}{\partial u} = 2tu - \lambda$$

c) Solving for u we obtain $u = \lambda/2t$ and thus

$$\dot{x} = -\lambda/2t$$

$$\dot{\lambda} = -2tx$$

$$x(1) = 1$$

$$\lambda(2) = 0$$

We can use Mathematica to obtain Bessel function as solution.

#2.

$$C[u] = \int_0^1 (x(t)^2 + x(t) + u(t)^2 + u(t))\,dt$$

$$\dot{x} = u$$

$$x(0) = 0$$

Solution:

(a) Defining $f(\varepsilon)$ by

$$f(\varepsilon) = \int_0^1 (x(t,\varepsilon)^2 + x(t,\varepsilon) + (u+\varepsilon\tilde{u})^2 + (u+\varepsilon\tilde{u}))\,dt + \int_0^1 [\lambda' x(t,\varepsilon) + \lambda \tfrac{\partial x}{\partial t}]\,dt$$
$$- \lambda(1)\,x(1,\varepsilon) + \lambda(0)\,x(0).$$

$$\rightarrow f'(0) = \int_0^1 (2x\tfrac{\partial x}{\partial \varepsilon} + \tfrac{\partial x}{\partial \varepsilon} + 2u\tilde{u} + \tilde{u} + \lambda' \tfrac{\partial x}{\partial \varepsilon} + \lambda\tilde{u})\,dt - \lambda(1)\tfrac{\partial x}{\partial \varepsilon}$$

$$\Rightarrow \quad \lambda' = -1 - 2x$$

$$0 = 2u + 1 + \lambda$$

$$\lambda(1) = 0$$

#3.

$$C[u] = \frac{1}{2}\int_0^1 \left[(x(t)-t-1)^2 + u(t)^2 \right] dt$$

$$\dot{x} = u$$

$$x(0) = 1.$$

Solution:

(a). Define

$$f(\varepsilon) = \frac{1}{2}\int_0^1 \left[(x(t,\varepsilon)-t-1)^2 + (u+\varepsilon\tilde{u})^2 \right] dt + \int_0^1 \left[\lambda' x(t,\varepsilon) + \lambda\cdot(u+\varepsilon\tilde{u}) \right] dt$$
$$- \lambda(1)x(1,\varepsilon) + \lambda(0)x(0,\varepsilon).$$

$$\Rightarrow f'(0) = \int_0^1 \left[(x-t-1)\frac{\partial x}{\partial \varepsilon} + u\tilde{u} \right] dt + \int_0^1 \left[\lambda' \frac{\partial x}{\partial \varepsilon} + \lambda\tilde{u} \right] dt - \lambda(1)\frac{\partial x}{\partial \varepsilon}.$$

$$\Rightarrow \lambda' = -x + t + 1$$

$$0 = u + \lambda$$

$$\lambda(1) = 0.$$

(b). The Hamiltonian is given by

$$H(x,u,\lambda) = \frac{1}{2}\left[(x-t-1)^2 + u^2 \right] + \lambda u$$

$$\dot{x} = u$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -x + t + 1$$

$$0 = \frac{\partial H}{\partial u} = u + \lambda$$

$$x(0) = 1$$

$$\lambda(1) = 0.$$

(b) The Hamiltonian is given by:

$$H(x,u,\lambda) = x^2 + x + u^2 + u + \lambda \cdot u$$

$$\Rightarrow \dot{x} = u$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -2x - 1$$

$$0 = \frac{\partial H}{\partial u} = 2u + 1 + \lambda.$$

(c) Solving for u we have:

$$u = \frac{-1-\lambda}{2}$$

$$\Rightarrow \dot{x} = \frac{-1-\lambda}{2}$$

$$\dot{\lambda} = -2x - 1$$

$$x(0) = 0$$

$$\lambda(1) = 0.$$

Differentiating we have that

$$\ddot{x} = -\frac{\dot{\lambda}}{2} = \frac{2x+1}{2} = x + 1/2$$

$$\Rightarrow x = A\cos(t) + B\sin(t) + t^2/4$$

$$x(0) = 0 \Rightarrow A = 0$$

$$\Rightarrow x = B\sin(t) + t^2/4$$

Therefore,

$$\dot{x} = B\cos(t) + t/2 = \frac{-1-\lambda}{2}$$

$$\Rightarrow 2B\cos(t) + t = -1 - \lambda$$

$$\Rightarrow \lambda = -1 - 2B\cos(t) - t$$

$$\lambda(1) = -1 - 2B\cos(1) - 1 = 0$$

$$\Rightarrow \boxed{B = \frac{-1}{\cos(1)}, \quad u = \frac{2B\cos(t) + t}{2}}$$

(c) Solving for u we have $\lambda = -u$ and thus:

$$\dot{x} = -\lambda$$

$$\dot{\lambda} = -x + t + 1$$

Therefore,

$$\ddot{\lambda} = -\dot{x} + 1$$

$$\Rightarrow \ddot{\lambda} = \lambda + 1$$

$$\Rightarrow \lambda = A\cos(t) + B\sin(t) + t^2/2$$

$$\Rightarrow \dot{\lambda} = -A\sin(t) + B\cos(t) + t = -x + t + 1$$

$$\Rightarrow x = A\sin(t) - B\cos(t) + 1$$

$$x(0) = 1 \Rightarrow B = 0$$

$$\lambda(1) = A\cos(1) + 1/2 = 0$$

$$\Rightarrow A = -1/2\cos(1)$$

Therefore,

$$\boxed{\begin{aligned} u &= -A\cos(t) - t^2/2 \\ u &= \frac{\cos(t) - t^2\cos(1)}{2} \end{aligned}}$$