MST 383/683
Homework #4

Due Date: October 15 2021

1. Consider an SIS network model with adjacency matrix A and where each node has a status $X_i(t) \in \{0, 1\}$ denoting whether node i is susceptible or infected at time t. Suppose further that the transition probabilities for the status of each node are given by:

$$
\begin{aligned}
P(X_i(t+\Delta t) = 1 | X_i(t) = 0) &= \beta \Delta t \sum_{j=1}^{n} A_{ij} X_j(t) \\
P(X_i(t+\Delta t) = 0 | X_i(t) = 0) &= 1 - \beta \Delta t \sum_{j=1}^{n} A_{ij} X_j(t) \\
P(X_i(t+\Delta t) = 0 | X_i(t) = 1) &= \alpha \Delta t \\
P(X_i(t+\Delta t) = 1 | X_i(t) = 1) &= 1 - \alpha \Delta t
\end{aligned}
$$

   (a) Prove that $[SS](t) + [SI](t) = \langle k_S(t) \rangle [S](t)$ and $[SI] + [II] = \langle k_I(t) \rangle [I](t)$, where $\langle k_S(t) \rangle$ and $\langle k_I(t) \rangle$ denote the average degree of the susceptible and infected nodes.

   (b) Prove that $[SSI] + [ISI] = (\langle k_S(t) \rangle - 1)\,[SI]$.

2. Consider an SIR network model with adjacency matrix A and where each node has a status $X_i(t) \in \{0, 1, 2\}$ denoting whether node i is susceptible, infected, or recovered at time t. Suppose further that the transition probabilities for the status of each node are given by:

$$
\begin{aligned}
P(X_i(t+\Delta t) = 0 | X_i(t) = 0) &= 1 - \beta \Delta t \sum_{j=1}^{n} A_{ij} X_j(t) \\
P(X_i(t+\Delta t) = 1 | X_i(t) = 0) &= \beta \Delta t \sum_{j=1}^{n} A_{ij} X_j(t) \\
P(X_i(t+\Delta t) = 2 | X_i(t) = 0) &= 0 \\
P(X_i(t+\Delta t) = 0 | X_i(t) = 1) &= 0 \\
P(X_i(t+\Delta t) = 1 | X_i(t) = 1) &= 1 - \alpha \Delta t \\
P(X_i(t+\Delta t) = 2 | X_i(t) = 1) &= \alpha \Delta t \\
P(X_i(t+\Delta t) = 0 | X_i(t) = 2) &= 0 \\
P(X_i(t+\Delta t) = 1 | X_i(t) = 2) &= 0 \\
P(X_i(t+\Delta t) = 2 | X_i(t) = 2) &= 1
\end{aligned}
$$

   Following the derivation we did for the SIS network model, show that

$$
\begin{aligned}
[\dot{S}] &= -\beta [SI], \\
[\dot{I}] &= \beta [SI] - \alpha [I], \\
[\dot{R}] &= \alpha [I].
\end{aligned}
$$

   Note, you just need to reproduce what we did in class in detail.

3. The differential equations for the dynamics of the nodes and edges for the SIS model are given by

$$\begin{aligned}
[\dot{S}] &= -\beta[SI] + \alpha[I] \\
[\dot{I}] &= \beta[SI] - \alpha[I] \\
[\dot{SS}] &= -2\beta[SSI] + 2\alpha[SI] \\
[\dot{SI}] &= \beta([SSI] - [ISI] - [SI]) + \alpha\left([II] - [SI]\right) \\
[\dot{II}] &= 2\beta([ISI] + [IS]) - 2\alpha[II]
\end{aligned}$$

(a) Using the following approximation:

$$[ABC] \approx \frac{\langle k\rangle - 1}{\langle k\rangle}\frac{[AB][BC]}{[B]},$$

derive a closed system of equations for the dynamics of the nodes and edges.

(b) Show that $[S] + [I]$ and $[SS] + 2[SI] + [II]$ are conserved quantities and thus

$$\begin{aligned}
n &= [S](0) + [I](0), \\
n\langle k\rangle &= [SS](0) + 2[SI](0) + [II](0)
\end{aligned}$$

are constant in time.

(c) Using the result of part (b), reduce the system of equations derived in part (a) to a system of three differential equations for $[S], [SS]$, and $[SI]$.

(d) Assuming further that $[SI] = \langle k\rangle[S] - [SS]$, reduce this system further to a system of two differential equations.

(e) For the system of differential equations you have derived, determine the fixed points, analyze their stability, and sketch phase portraits that illustrate all of the qualitatively different cases that occur.

(f) Calculate a dimensionless parameter $\mathcal{R}_0$ such that if $\mathcal{R}_0 > 1$ the disease becomes endemic.

4. For the SIR model defined in problem #2, derive differential equations for the edges:

$$[SS], [SI], [SR], [II], [IR], [RR].$$

5. In this problem you will derive differential equations for an SIS model in which edges can be paused. Specifically, individuals will pause connections at a rate proportional to the total number of infections. This could model a disease in which infected individuals are asymptomatic but as more infections are reported individuals remove themselves from the network.

(a) For the SIS model given in problem #3, introduce a new class of edges $\overline{[SS]}$, $\overline{[IS]}$, and $\overline{[II]}$ which denote paused connections. Develop a system of eight differential equations for the dynamics of $[S], [I], [SS], [SI], [II], \overline{[SS]}, \overline{[SI]}$ and $\overline{[II]}$ in which paused connections are introduced at rate proportional to the number of infections. Note, your equations should be conservative in the sense that in addition to $[S] + [I]$, $[SS] + 2[SI] + [II] + \overline{[SS]} + 2\overline{[SI]} + \overline{[II]}$ is a conserved quantity.

(b) Using the same moment closure approximation defined in problem #3(a), close this system of equations.

(c) Using the conserved quantities reduce this system of equations to a set of six differential equations.

(d) Calculate a dimensionless parameter $\mathcal{R}_0$ such that if $\mathcal{R}_0 > 1$ the disease becomes endemic.