Lecture 10: Next Generation Approach

The Jacobian can be too difficult to analyze directly sometimes. The idea of the next generation approach is to track infections.

Idea:

1. split equation into infected and noninfected components:

$$\dot{x}_i = f_i(x,y) \quad i=1,\dots,n$$
$$\dot{y}_j = g_j(x,y) \quad j=1,\dots,n$$

x ~ infected compartments

y ~ uninfected compartments

1. $\dot{S} = \Lambda - \beta I S - \nu S$
$\dot{E} = \beta I S - \gamma E - \nu E$
$\dot{I} = \gamma E - \alpha I - \nu I$
$\dot{R} = \alpha I - \nu R$

$\downarrow$

$\dot{E} = \beta I S - \gamma E - \nu E$
$\dot{I} = \gamma E - \alpha I - \nu I$

Infected Compartments

$\dot{S} = \Lambda - \beta I S - \nu S$
$\dot{R} = \alpha I - \nu R$

Uninfected Compartments

2. Split infected compartment into new infections minus old infections

$$\dot{x}_i = \mathcal{F}_i(x,y) - \mathcal{V}_i(x,y)$$
$$\dot{y}_j = g_j(x,y)$$

a) $\mathcal{F}_i(0,y) = 0$, $\mathcal{V}_i(0,y) = 0$

b) $\mathcal{F}_i(x,y) \geq 0$

c) $\mathcal{V}_i(x,y) \leq 0$ when $x_i = 0$

d) $\sum_i \mathcal{V}_i(x,y) \geq 0$

$\mathcal{F}_1(E,I,S,R) = \beta I S$
$\mathcal{F}_2(E,I,S,R) = \gamma E$
$\mathcal{V}_1(E,I,S,R) = -\gamma E - \nu E$
$\mathcal{V}_2(E,I,S,R) = -\alpha I - \nu I$

All properties are satisfied in this case.

3. There exists a disease free equilibrium $\mathcal{E}_0=(0,y_0)$ such that all initial conditions of the form $(0,y)$ approach $(0,y_0)$ as $t\to\infty$.

Disease free equilibrium for SEIR is

$(0,0,\Lambda/\nu,0)$.

If we have an initial condition of the form $(0,0,S_0,0)$ then

$$\dot{E}=0$$
$$\dot{I}=0$$
$$\dot{S}=\Lambda-\nu S$$
$$\dot{R}=0$$
$$\Rightarrow \lim_{t\to\infty} S(t)=\Lambda/\nu$$

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4. By construction if we linearize near the disease free equilibrium:

$$\Rightarrow \dot{x}_i=\mathcal{F}_i(x,y)-\mathcal{V}_i(x,y)$$
$$\dot{y}_j=g_j(x,y)$$

$$\Rightarrow \dot{x}_i=\sum_{j=1}^{n}\left(\frac{\partial \mathcal{F}_i}{\partial x_j}\Big|_{(0,y_0)}x_j-\frac{\partial \mathcal{V}_i}{\partial x_j}\Big|_{(0,y_0)}x_j\right),$$

where by property (a) the other partial derivatives vanish.

Letting $F=\frac{\partial \mathcal{F}_i}{\partial x_j}$, $V=\frac{\partial \mathcal{V}_i}{\partial x_j}$, we obtain

$$\dot{\vec{x}}=(F-V)\vec{x},$$

near $(0,y_0)$.

For the SEIR model we obtain

- $\frac{\partial \dot{E}}{\partial E}\Big|_{(0,y_0)}=-\eta-\nu=-\eta-\nu$
- $\frac{\partial \dot{E}}{\partial I}\Big|_{(0,y_0)}=\beta S\Big|_{S=\Lambda/\nu_0}=\frac{\beta\Lambda}{\nu_0}$
- $\frac{\partial \dot{I}}{\partial E}=\eta$
- $\frac{\partial \dot{I}}{\partial I}=-\alpha-\nu$

$$\Rightarrow F=\begin{bmatrix}0 & \beta\Lambda/\nu_0\\ \eta & 0\end{bmatrix}$$

$$V=\begin{bmatrix}\eta+\nu & 0\\ 0 & \alpha+\nu\end{bmatrix}$$

5. The next generation matrix is defined as

$$K = FV^{-1}$$

and

$$\mathcal{R}_0 = \rho(FV^{-1}),$$

i.e. the maximum of the absolute values of the eigenvalues of FV^{-1}.

$$F\cdot V^{-1} = \begin{bmatrix} 0 & \beta\Delta/\omega \\ \eta & 0 \end{bmatrix}\begin{bmatrix} 1/(\eta+\omega) & 0 \\ 0 & 1/(\alpha+\omega) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \beta\Delta/\omega(\alpha+\omega) \\ \eta/(\eta+\omega) & 0 \end{bmatrix}$$

$$\Rightarrow \lambda_{1,2} = \pm\sqrt{\frac{\beta\Delta\eta}{\omega(\alpha+\omega)(\eta+\omega)}}$$

The next generation reproduction number is

$$\mathcal{R}_0 = \sqrt{\frac{\beta\Delta\eta}{\omega(\alpha+\omega)(\eta+\omega)}}.$$

Why does this work??

Let $m(F-V)$ = max real part of eigenvalues of $F-V$.

FV^{-1} - has nonnegative entries $\Rightarrow \mathcal{R}_0$ is an eigenvalue of $F\cdot V^{-1}$.

Theorem:

$m(F-V)<0$ if and only if $\rho(FV^{-1})<1$

$m(F-V)>0$ if and only if $\rho(FV^{-1})>1$

proof (Idea):

(totally wrong):

$F-V<0$	$F-V>0$
$\Rightarrow F<V$	$F>V$
$\Rightarrow F\cdot V^{-1}<1$	$F\cdot V^{-1}>1$

Example:

$\dot{S} = -\beta S I$

$\dot{I} = \beta S I - \alpha I$

$\dot{R} = \alpha I$

Infected compartments

$\dot{I} = \beta S I - \alpha I$

Disease free state!

$S = N$

$I = 0$

$R = 0$

$\frac{\partial \dot{I}}{\partial I} = \beta S - \alpha$

$\left.\frac{\partial \dot{I}}{\partial I}\right|_{S=N} = \beta N - \alpha$

$\Rightarrow F = \beta N$

$V = \alpha$

$\rightarrow F \cdot V^{-1} = \frac{\beta N}{\alpha}.$

Jacobian at $(N, 0, 0)$

$$= \begin{bmatrix} 0 & -\beta N & 0 \\ 0 & \beta N - \alpha & 0 \\ 0 & \alpha & 0 \end{bmatrix}$$

$\Rightarrow$ Eigenvalues are $\beta N - \alpha, 0, 0.$

$\Rightarrow R_0 = \frac{\beta N}{\alpha}.$

What is going on with 0 eigenvalues

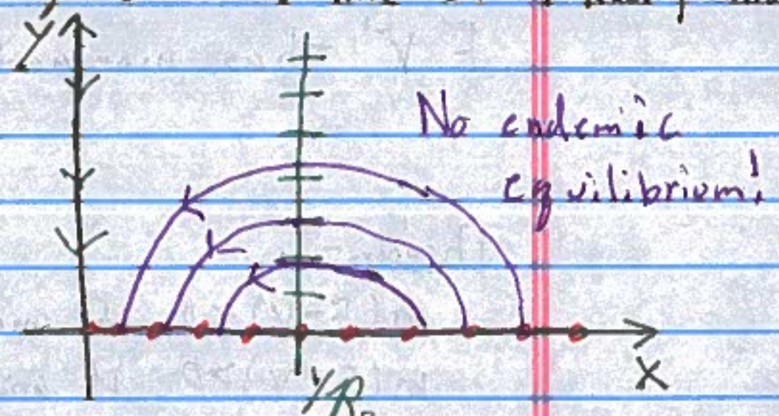
Reduce to dimensionless form!

$\dot{x} = -R_0 x y$

$\dot{y} = R_0 x y - y$

$x = 0$, $x = 1/R_0$ are nullclines

$y = 0$ is a line of fixed points.



Numerical Computation.

Suppose a matrix A has only positive eigenvalues. How can we compute the largest eigenvector??

Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be an eigenbasis with eigenvalues $\lambda_1, \dots, \lambda_n$ ordered so that $\lambda_1 > \lambda_2 > \lambda_3 > \dots > \lambda_n$.

$$\vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$$

$$\Rightarrow A\vec{x} = c_1 \lambda_1 \vec{v}_1 + \dots + c_n \lambda_n \vec{v}_n$$

$$A^2 \vec{x} = c_1 \lambda_1^2 \vec{v}_1 + \dots + c_n \lambda_n^2 \vec{v}_n$$

$$\vdots$$

$$A^m \vec{x} = c_1 \lambda_1^m \vec{v}_1 + \dots + c_n \lambda_n^m \vec{v}_n$$

$$= c_1 \lambda_1^m \left(\vec{v}_1 + \frac{\lambda_2^m}{\lambda_1^m} \vec{v}_2 + \dots + c_n \frac{\lambda_n^m}{\lambda_1^m} \vec{v}_n \right)$$

$$\approx c_1 \lambda_1^m \vec{v}_1$$

Therefore,

$$\|A^m x\| \approx c_1 \lambda_1^m$$

$$\Rightarrow \vec{v}_1 \approx \frac{A^m \vec{x}}{\|A^m \vec{x}\|}$$

To find λ_1, note that

$$\lambda_1 = \frac{\vec{v}_1^T A \vec{v}_1}{\|\vec{v}_1\|} = v_1^T A \vec{v}_1$$