Lecture 13: Basics of Optimal Control Theory

Example (SIS with treatment):

$$\dot{S} = \nu N - \beta S I + \alpha I - \nu S + u(t) I, \quad S(0) = S_0$$
$$\dot{I} = \beta S I - \alpha I - \nu I - u(t) I, \quad I(0) = I_0$$

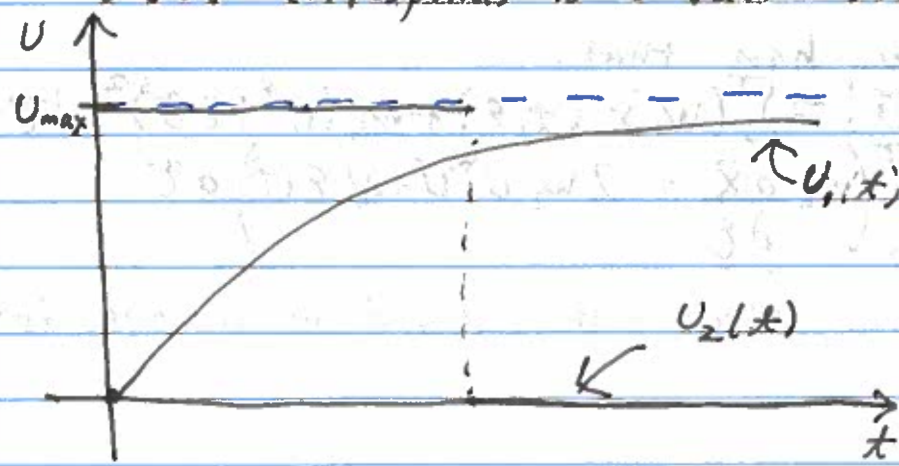
where $N = S + I$. $\Rightarrow \dot{I} = \beta(N - I) - \alpha I - \nu I - u(t) I$

- ν birth/death rate $\Rightarrow \frac{dx}{d\tau} = R_0(1-x) - x - Ax - u(t)x$
- α recovery rate
- u treatment rate (function of time)
- β force of infection

$$\mathcal{A} = \{ v : [0,T] \to [0, U_{max}] : v \text{ is piecewise smooth} \}.$$

We assume $u \in \mathcal{A}$. (admissible set of controls).

- $u \in \mathcal{A}$ corresponds to a vacine strategy.



Cost of treatment:

$$C[u] = \int_0^T (w_1 x(\tau) + w_2 u^2(\tau))\, d\tau, \quad w_1, w_2 > 0 \text{ are weights.}$$

Goal:

Find $u^* \in \mathcal{A}$ such that

$$C[u^*] = \min_{u \in \mathcal{A}} C[u].$$

Optimization Problem:

- $$\min_{u \in A^*} \int_0^T \underbrace{(w_1 x(\tau) + w_2 u^2(\tau))}_{L(x,u) = \text{Lagrangian.}} d\tau$$
- Subject to: $x(0) = x_0$
$$\frac{dx}{d\tau} = g(\tau, u(\tau))$$

Necessary Conditions:

Assume u^* is a minimizer. Let $\tilde{u} \in A$ and define:
$$F(\varepsilon) = C[u^* + \varepsilon \tilde{u}], \quad F: \mathbb{R} \longmapsto \mathbb{R}.$$
$$F(0) = C[u^*] \text{ is a minimum.}$$

Assuming differentiability we know that
$$F'(0) = 0$$

Calculating we have that:
$$C[u^* + \varepsilon \tilde{u}] = \int_0^T (w_1 x(t, \varepsilon) + w_2 (u^* + \varepsilon \tilde{u})^2) d\tau$$
$$\frac{dF}{d\varepsilon} = \int_0^T \left(w_1 \underbrace{\frac{\partial x}{\partial \varepsilon}} + 2 w_2 u^* \tilde{u} + w_2 \varepsilon \tilde{u}^2 \right) d\tau$$

What is this? How do we compute it?

Note
$$\int_0^T \frac{d}{d\tau} [\lambda(\tau) x(\tau, \varepsilon)] d\tau = \lambda(T) x(T, \varepsilon) - \lambda(0) x(0, \varepsilon)$$
$$= \lambda(T) x(T, \varepsilon) - \lambda(0) x_0.$$

$$\Rightarrow F(\varepsilon) = C[u^* + \varepsilon \tilde{u}] + \underbrace{\int_0^T \left[\lambda' x(\tau, \varepsilon) + \lambda(\tau) \frac{\partial x}{\partial \tau}\right] d\tau - \lambda(T) x(T, \varepsilon) + \lambda(0) x_0}_{\text{Null-Lagrangian}}$$

$$\Rightarrow F(\varepsilon) = C[u^* + \varepsilon \tilde{u}] + \int_0^T [\lambda' x(\tau, \varepsilon) + \lambda(\tau) g(x, u^* + \varepsilon \tilde{u})] d\tau - \lambda(T) x(T, \varepsilon) + \lambda(0) x_0$$

$$\frac{dE}{d\varepsilon} = \int_0^T \left(w_1 \frac{\partial x}{\partial \varepsilon} + 2w_2(u^*\tilde{u} + \tilde{u}^2\varepsilon) + \lambda' \frac{\partial x}{\partial \varepsilon} + \lambda \frac{\partial g}{\partial u}(u^* + \varepsilon\tilde{u})\tilde{u} \right) d\tau$$

$$- \lambda(T) \frac{\partial x}{\partial \varepsilon}.$$

$$\Rightarrow \left.\frac{dE}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^T \left[\left(w_1 + \lambda' \right) \frac{\partial x}{\partial \varepsilon} + \left(2w_2 u^* + \lambda \frac{\partial g}{\partial u}(u^*) \right) \tilde{u} \right] - \lambda(T) \frac{\partial x}{\partial \varepsilon}$$

$$= \int_0^T \left[\left(w_1 + \lambda' \right) \frac{\partial x}{\partial \varepsilon} + \left(2w_2 u^* + \lambda(-x) \right) \tilde{u} \right] - \lambda(T) \frac{\partial x}{\partial \varepsilon}$$

For this to be equal to zero we obtain the following necessary conditions:

$$\boxed{\begin{aligned} &\lambda' = -w_1 \rightarrow \text{Adjoint equation} \\ &u^* = \frac{\lambda x}{2w_2} \\ &\lambda(T) = 0 \rightarrow \text{Transversality condition} \\ &x(0) = x_0 \\ &\frac{dx}{d\tau} = R_0(1-x) - x - Ax - u^* x \end{aligned}}$$

$$\Downarrow$$

$$\lambda = -w_1 t + w_1 T = -w_1(t - T)$$

$$\frac{dx}{d\tau} = R_0(1-x) - x - Ax + \frac{w_1(t-T)}{2w_2} x^2$$

$$x(0) = x_0$$

$$u^*(t) = \frac{w_1(T-t)}{2w_2} x.$$

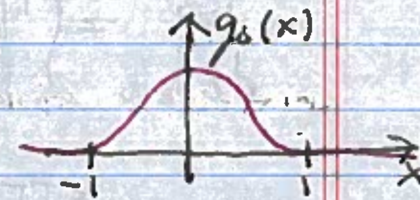
$\rightarrow$ treat early

$\rightarrow$ treat when infections are high.

Theorem – If $f \in C^1([a,b];\mathbb{R})$ and if
$$\int_a^b f(x)\,g(x)\,dx = 0$$
for all $g \in C^\infty([a,b];\mathbb{R})$ then $f = 0$.

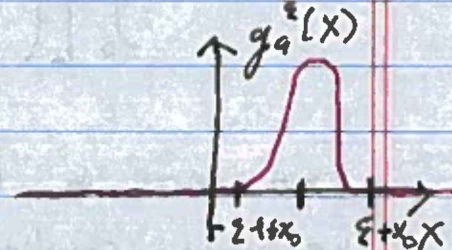
proof:

1. Let $g_0(x) = \begin{cases} \frac{1}{c}\exp\left(-\frac{1}{x^2-1}\right), & |x|<1 \\ 0, & |x| \geq 1 \end{cases}$



where
$$c = \int_{-1}^{1} \exp\left(-\frac{1}{x^2-1}\right)dx.$$

Define
$$g_{x_0}^{\varepsilon}(x) = \frac{1}{\varepsilon}\, g_0\left(\frac{x-a}{\varepsilon}\right).$$



By construction:
$$\int_a^b g_{x_0}^{\varepsilon}(x)\,dx = 1$$
if $x_0+\varepsilon < b$ and $x_0 - \varepsilon > a$. We now claim that:
$$\lim_{\varepsilon \to 0} \int_a^b f(x)\, g_{x_0}^{\varepsilon}(x)\,dx = f(x_0).$$

To show this, we have that:
$$\left|\int_a^b f(x)\, g_{x_0}^{\varepsilon}(x)\,dx - f(x_0)\right| = \left|\int_a^b f(x)\, g_{x_0}^{\varepsilon}(x)\,dx - \int_a^b f(x_0)\, g_{x_0}^{\varepsilon}(x)\,dx\right|$$
$$= \left|\int_a^b \big(f(x)-f(x_0)\big)\, g_{x_0}^{\varepsilon}(x)\,dx\right|$$
$$\leq \int_a^b |f(x)-f(x_0)|\, g_{x_0}^{\varepsilon}(x)\,dx$$
$$= \int_a^b |f'(c)|\cdot|x-x_0|\, g_{x_0}^{\varepsilon}(x)\,dx$$
$$= \int_{x_0-\varepsilon}^{x_0+\varepsilon} |f'(c)|\cdot|x-x_0|\, g_{x_0}^{\varepsilon}(x)\,dx$$

$$\Rightarrow \left| \int_a^b f(x) g_{x_0}^{\varepsilon}(x)dx - f(x_0) \right| \le \int_{x_0-\varepsilon}^{x_0+\varepsilon} M \cdot \varepsilon\, g_{x_0}^{\varepsilon}(x)\,dx$$
$$= M \cdot \varepsilon.$$

Consequently,
$$\lim_{\varepsilon \to 0} \int_a^b f(x) g_{x_0}^{\varepsilon}(x)dx = f(x_0).$$

2. Now, by assumption:
$$\int_a^b f(x)\, g_{x_0}^{\varepsilon}(x)dx = 0$$
$$\Rightarrow \lim_{\varepsilon \to 0} \int_a^b f(x) g_{x_0}^{\varepsilon}(x)dx = \lim_{\varepsilon \to 0} 0$$

$$\Rightarrow f(x_0) = 0.$$

Since this is true for all x_0 it follows that $f = 0$. ■

General Theory:
$$\dot{x} = f(t, x(t), u(t))$$
$$x(0) = x_0$$
$$u \in \mathcal{A}$$
$$C[u] = \int_0^T L(t, x(t,u), u)\, t$$

Suppose $u^* \in \mathcal{A}$ minimizes C. Define,
$$F(\varepsilon) = C[u^* + \varepsilon \tilde{u}],$$
where $\tilde{u} \in \mathcal{A}$. Now,
$$F(\varepsilon) = \int_0^T L(t, x(t, u^* + \varepsilon \tilde{u}), u^* + \varepsilon \tilde{u})\,dt$$
$$= \int_0^T L(t, x(t, \varepsilon), u^* + \varepsilon \tilde{u})\,dt$$
$$\Rightarrow F'(\varepsilon) = \int_0^T \left(\frac{\partial L}{\partial x} \frac{\partial x}{\partial \varepsilon} + \left.\frac{\partial L}{\partial u}\right|_{u^* + \varepsilon \tilde{u}} \tilde{u} \right) dt$$

We also have that

$$F(\varepsilon) = \int_0^T L(t, x(t,\varepsilon), u^* + \varepsilon\tilde{u})\,dt + \int_0^T \frac{d}{dt}[\lambda(t)x(t,\varepsilon)]\,dt - \lambda(T)x(T,\varepsilon) + \lambda(0)x_0$$

$$\Rightarrow \frac{d}{d\varepsilon}F(\varepsilon) = \int_0^L \left(\frac{\partial L}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial L}{\partial u}\Big|_{u^*+\varepsilon\tilde{u}}\tilde{u}\right)dt + \int_0^T \left[\lambda'\frac{\partial x(t,\varepsilon)}{\partial \varepsilon} + \lambda\frac{\partial^2 x}{\partial\varepsilon\partial t}\right]dt - \lambda(T)\frac{\partial x}{\partial \varepsilon}.$$

$$= \int_0^L \left(\frac{\partial L}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial L}{\partial u}\Big|_{u^*+\varepsilon\tilde{u}}\tilde{u}\right)dt + \int_0^T \left[\lambda'\frac{\partial x}{\partial \varepsilon} + \lambda\frac{\partial}{\partial \varepsilon}f(t,x,u^*+\varepsilon\tilde{u})\right]dt - \lambda(T)\frac{\partial x}{\partial \varepsilon}$$

$$+\lambda\frac{\partial f}{\partial x}\frac{\partial x}{\partial \varepsilon}.$$

$$\Rightarrow \frac{d}{d\varepsilon}F(\varepsilon) = \int_0^L \left(\frac{\partial L}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial L}{\partial u}\Big|_{u^*+\varepsilon\tilde{u}}\tilde{u}\right)dt + \int_0^T \left[\lambda'\frac{\partial x}{\partial \varepsilon} + \lambda\frac{\partial f}{\partial u}\Big|_{u^*+\varepsilon\tilde{u}}\tilde{u}\right]dt - \lambda(T)\frac{\partial x}{\partial \varepsilon}$$

$$\Rightarrow F'(0) = \int_0^L \left[\left(\frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} + \lambda'\right)\frac{\partial x}{\partial \varepsilon} + \left(\frac{\partial L}{\partial u} + \lambda\frac{\partial f}{\partial u}\right)\tilde{u}\right]dt - \lambda(T)\frac{\partial x}{\partial \varepsilon}$$

$$\Downarrow$$

- $\frac{\partial L}{\partial u} + \lambda\frac{\partial f}{\partial u} = 0$ (optimality condition)
- $\lambda' = -\frac{\partial L}{\partial x} - \lambda\frac{\partial f}{\partial x}$ (adjoint Equations)
- $\lambda(T) = 0$ (transversality condition)
- $\dot{x} = f(t,x,u)$ (dynamics)
- $x(0) = x_0$ (initial condition).

Hamiltonian:

$$\dot{x} = f(t, x(t), u(t))$$
$$x(0) = x_0$$
$$C[u] = \int_0^T L(t, x, u)\,dt$$

Define

$$\boxed{H = L(t, x, u) + \lambda f(t, x, u)}$$

- $\frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda \frac{\partial f}{\partial u} = 0$
- $\dot{\lambda} = -\frac{\partial H}{\partial x} = -\left(\frac{\partial L}{\partial x} + \lambda \frac{\partial f}{\partial x}\right)$
- $\lambda(T) = 0$
- $\dot{x} = \frac{\partial H}{\partial \lambda} = f$

Example:

$$\min_u \frac{1}{2}\int_0^1 (3x(t)^2 + u(t)^2)\,dt$$

$$\dot{x} = x(t) + u(t)$$
$$x(0) = 1.$$

$$\Rightarrow H = L + \lambda f$$
$$= (3x^2 + u^2) + \lambda(x + u)$$

$$\Rightarrow \frac{\partial H}{\partial u} = 2u + \lambda = 0$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -6x - \lambda, \quad \lambda(T) = 0$$

$$\dot{x} = \frac{\partial H}{\partial \lambda} = x + u.$$

$\Rightarrow u = -\lambda/2$

$$\Rightarrow \boxed{\begin{aligned} \dot{x} &= x - \lambda/2 \\ \dot{\lambda} &= -6x + \lambda \\ x(0) &= 1 \\ \lambda(T) &= 0 \end{aligned}}$$

How do we solve O.D.E?

$$\dot{\vec{x}} = \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ -6 & -1 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = A\vec{x}$$

Solutions are of the form

$$\vec{x} = c_1 e^{\mu_1 t}\vec{v}_1 + c_2 e^{\mu_2 t}\vec{v}_2,$$

where μ_i is an eigenvalue with corresponding eigenvector $\vec{v}_i$.

$$\mu = \frac{\pm\sqrt{-4(-4)}}{2} = \pm 2.$$

$$\Rightarrow \vec{x}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow x(t) = c_1 e^{2t} + c_2 e^{-2t}$$
$$\lambda(t) = -c_1 e^{2t} + 3c_2 e^{-2t}$$

$$x(0) = c_1 + c_2 = 1$$
$$\lambda(1) = -c_1 e^{2} + 3c_2 e^{-2} = 0$$
$$\Rightarrow c_2 = \tfrac{1}{3} c_1 e^4$$

$$\Rightarrow c_1 + \tfrac{1}{3} c_1 e^4 = 1$$
$$\Rightarrow c_1 = \frac{1}{1 + \frac{1}{3}e^4} = \frac{3}{3 + e^4}$$
$$c_2 = \frac{e^4}{3 + e^4}$$

$$\Rightarrow \boxed{\begin{aligned} x(t) &= \frac{3}{3+e^4}e^{2t} + \frac{e^4}{3+e^4}e^{-2t} \\ u(t) &= -\frac{\lambda}{2} = \frac{3}{2}\,\frac{1}{3+e^4}e^{2t} - \frac{3}{2}\,\frac{e^4}{3+e^4}e^{-2t} \end{aligned}}$$