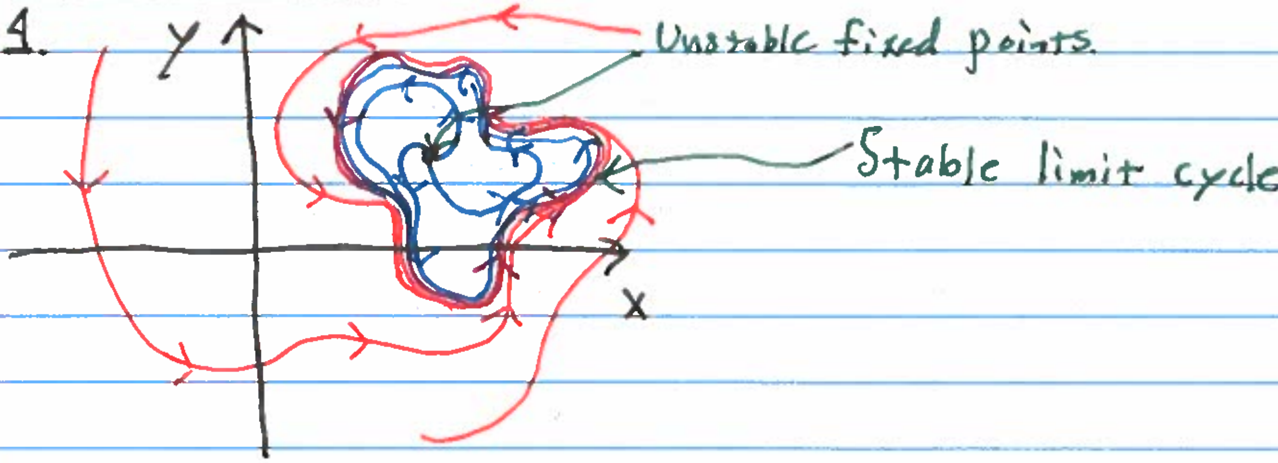
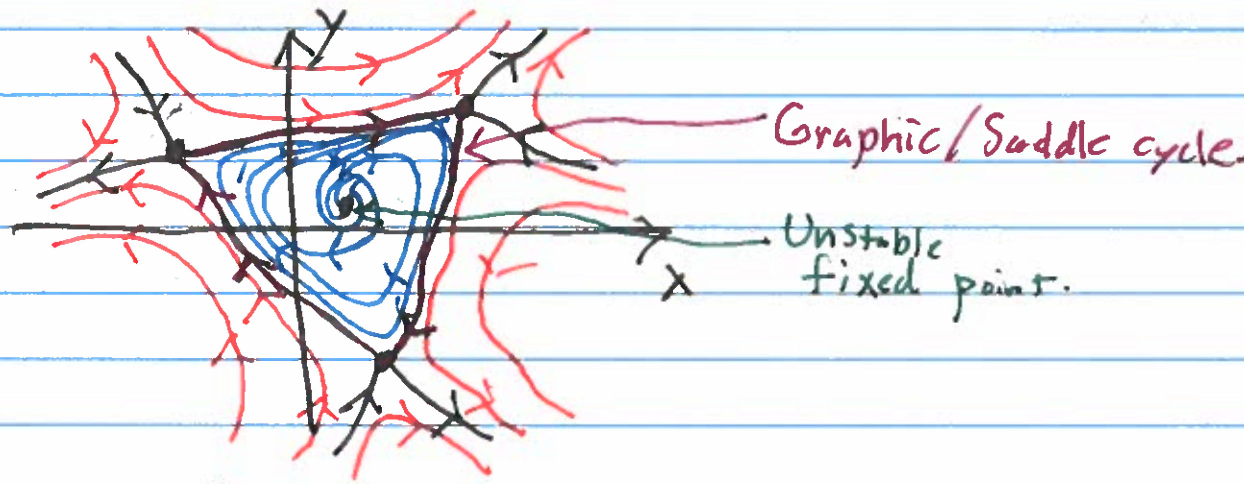
Lecture 5: Global Stability

Exotic Phase Portraits:

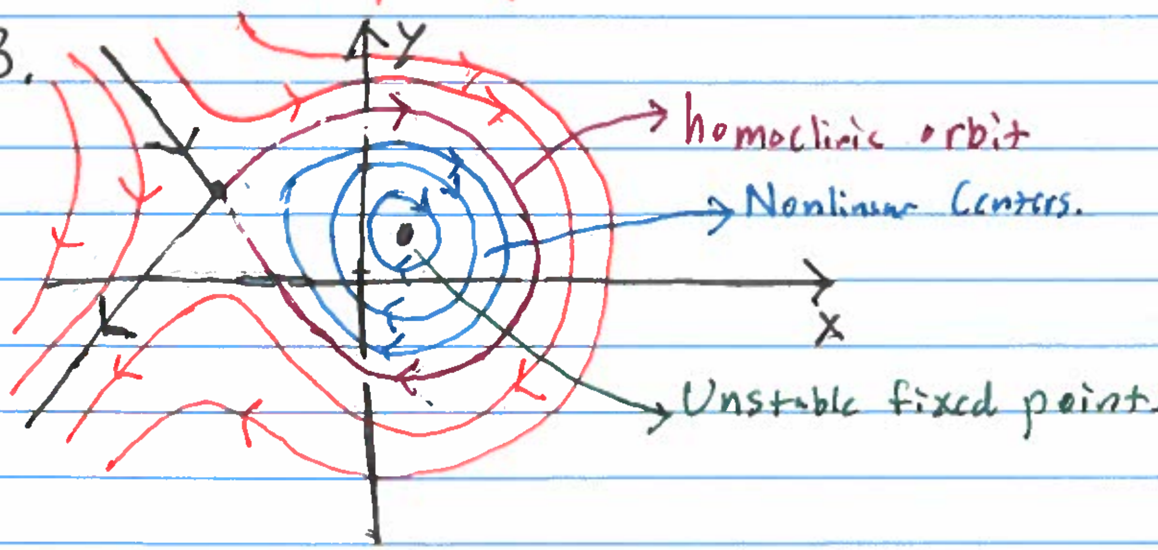
1.



2.



3.



Theorem: (Poincaré Bendixon Theorem) - Suppose $D \subseteq \mathbb{R}^2$ is a trapping region with finitely many equilibria for the system

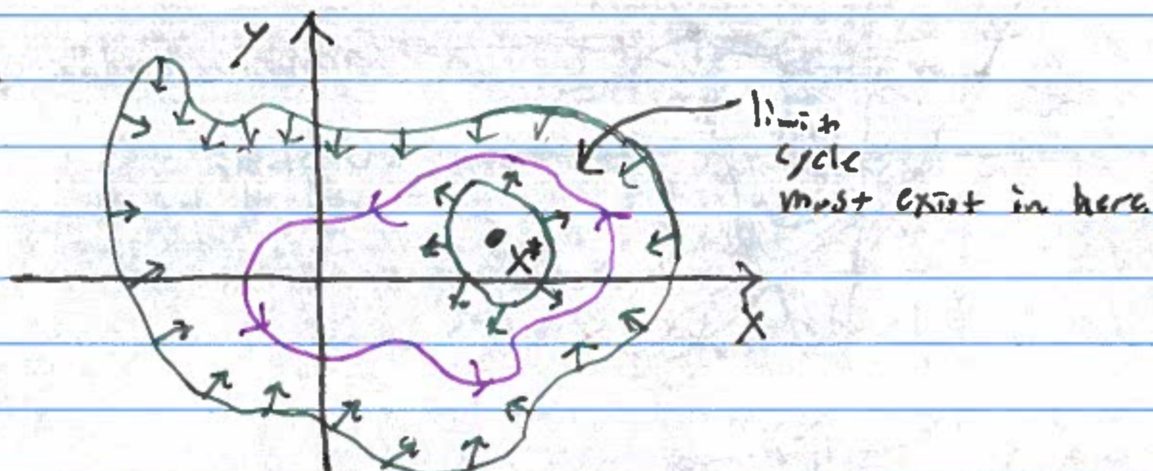
$$\dot{x} = F(x),$$

i.e. flow is pointing inward on ∂D, i.e. $F(\vec{x}) \cdot \vec{n} < 0$ on $\partial \Omega$, then inside D one of the following holds

1. $\lim_{t \to \infty} x(t) = x^*$ a fixed point
2. $x(t)$ approaches a limit cycle
3. $x(t)$ approaches a graphic.

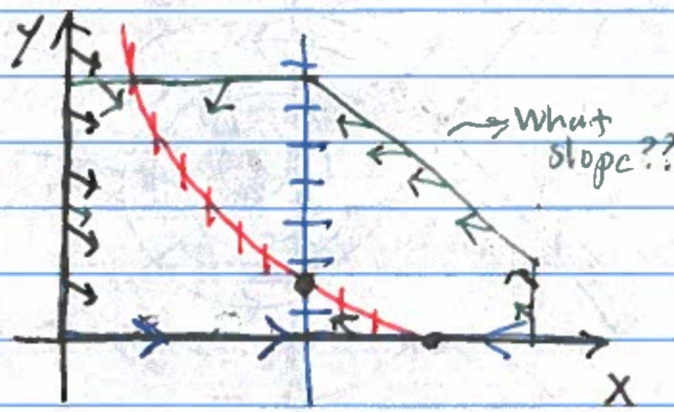
Example:

1.



2. $\dfrac{dx}{d\tau} = \varsigma(1-x) - Axy$

$\dfrac{dy}{d\tau} = y(Ax - (1+s))$



$$\frac{dy}{dx} = \frac{s(1-x) - Axy}{y(Ax-(1+s))}$$

For large y:

$$\frac{dy}{dx} \approx \frac{-Axy}{y(Ax-(1+s))} = \frac{-Ax}{Ax-(1+s)}$$

Since $Ax-(1+s)>0$ it follows that

$$\frac{dy}{dx} \lesssim \frac{-Ax}{Ax} = -1$$

For large enough y:

$$\frac{dy}{dx} < -2,$$

Thus if we pick a slope of -2 we can guarantee a trapping region.

$\Rightarrow$ Trajectories must go to fixed point or limit cycle!

<u>Dulac Criterion</u> - Let $D \subseteq \mathbb{R}^2$ be an open simply connected domain. Assume $f, g: D \to \mathbb{R}$ be continuously differentiable functions. Assume there exists a function $\mathcal{D}: D \to \mathbb{R}$ such that

$$\nabla \cdot (\mathcal{D} F) = \frac{\partial(\mathcal{D} f)}{\partial x} + \frac{\partial(\mathcal{D} g)}{\partial y}$$

is either strictly positive or negative then

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

cannot have a limit cycle in D.

proof

Let C be a limit cycle bounding a region A.

$$0 \neq \iint_A \nabla\cdot(D\cdot\frac{dx}{d\tau})\,dA = \oint_C D\,\frac{dx}{d\tau}\cdot\vec{n}\,dl = 0$$

which is a contradiction.

Example:

$$\frac{dx}{d\tau} = \varsigma(1-x) - Axy$$

$$\frac{dy}{d\tau} = y(Ax-(1+\varsigma))$$

$$\nabla\cdot F = \frac{\partial}{\partial x}(\varsigma(1-x)-Axy) + \frac{\partial}{\partial y}(y(Ax-(1+\varsigma)))$$

$$= -\varsigma - Ay + Ax - 1 - \varsigma$$

This does not work!!

$$\nabla\cdot(\tfrac{1}{y}F) = \frac{\partial}{\partial x}\left(\frac{\varsigma(1-x)-Axy}{y}\right) + \frac{\partial}{\partial y}\left(Ax-(1+\varsigma)\right)$$

$$= \frac{1}{y}(-\varsigma - Ay) < 0 \text{ in first quadrant.}$$

$\Rightarrow$ No limit cycles in first quadrant!!