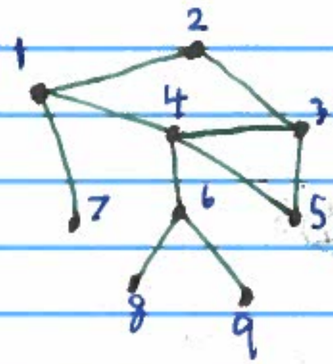
# Lecture 6: Epidemics on Static Networks.

## Intro to Networks

- A network (graph) consists of nodes and edges, also called vertices and links.




$V = \{1, \ldots, 9\}$
$\downarrow$
Set of nodes:

$e = \{12, 14, 17, 23, 34, 35, 45, 46, 68, 69$
$21, 41, 71, 32, 43, 53, 54, 64, 86, 96\}$
$\searrow$ Set of edges

- Adjacency matrix encodes connections

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix} \end{array} = A.$$

- degree(node) = # of edges coming out of node.

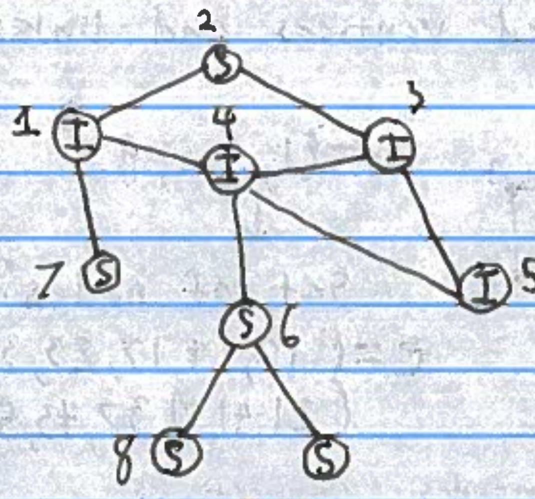
$$\boxed{\begin{aligned} \text{degree}(i) &= \sum_{j=1} A_{ij} \\ \langle \text{degree} \rangle &= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \end{aligned}}$$

$$K_j = \sum_{i=1}^{n} A_{ij}$$
$\uparrow$
degree of node $j$.

## SIS on a Network

Given a network $(V, E)$ with adjacency matrix $A$.

$\vec{I}$ = vector of 1's and 0's with infection states of node $i$



$$\vec{I}(0) = \begin{bmatrix} I_1(0) \\ \vdots \\ I_9(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P(I_i(t+\Delta t)=0 \mid I_i(t)=1) = \alpha \Delta t$$
$$P(I_i(t+\Delta t)=1 \mid I_i(t)=1) = 1-\alpha \Delta t$$

Recovery Probabilities.

$$P(I_i(t+\Delta t)=1 \mid I_i(t)=0) = \beta \Delta t \sum_{j=1}^{n} A_{ij} I_j(t)$$
$$P(I_i(t+\Delta t)=0 \mid I_i(t)=0) = 1-\beta \Delta t \sum_{j=1}^{n} A_{ij} I_j(t).$$

Infection probabilities

This is an example of a Markhov chain with a very complicated transition matrix.

## Expected Values

On average what is the network dynamics??

$$[I](t) = \sum_{i=1}^{n} P(I_i(t)=1)$$

$$[S](t) = \sum_{i=1}^{n} P(I_i(t)=0)$$

$$[II](t) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} P(I_i(t)=1, I_j(t)=1)$$

$$[IS](t) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} P(I_i(t)=1, I_j(t)=0)$$

$$[SS](t) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} P(I_i(t)=0, I_j(t)=0)$$

**Theorem** - $[I](t) + [S](t) = n$.

*proof:*

$$[I](t) + [S](t) = \sum_{i=1}^{n}\left(P(I_i(t)=1) + P(I_i(t)=0)\right)$$

$$= \sum_{i=1}^{n}\left( P(I_i(t)=1) + 1 - P(I_i(t)=0)\right)$$

$$= n.$$

■

**Theorem** - $[II](t) + [IS](t) + [SS](t) + [SI](t) = n\langle k\rangle$.

*proof:*

$$[II](t) + [IS](t) + [SI](t) + [SS](t)$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} A_{ij}\left(P(I_i(t)=1, I_j(t)=1) + \ldots + P(I_i(t)=0, I_j(t)=0)\right)$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} A_{ij}$$

$$= n\langle K\rangle.$$

■

**Theorem** - $[SI](t) = [IS](t)$.

*proof:*

$$[SI](t) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\left(P(I_i = 0, I_j = 1)\right)$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} A_{ij}\, P(I_i = 0, I_j = 1)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\, P(I_i = 0, I_j = 1)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ji}\, P(I_j = 1, I_i = 0)$$

$$= [IS](t).$$

**Definition** - $[ABC] = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} g_{ij}\, g_{jk}\, P(I_i = A, I_j = B, I_k = C).$

Average number of triple links.

**Theorem** - $[\dot{S}] = -\beta[IS] + \alpha[I]$

$[\dot{I}] = \beta[IS] - \alpha[I]$

*proof:*

$$[I](t+\Delta t) = \sum_{i=1}^{n} P(I_i(t+\Delta t) = 1)$$

$$= \sum_{i=1}^{n}\Big(P(I_i(t+\Delta t) = 1 \mid I_i(t) = 1)\, P(I_i(t) = 1) + P(I_i(t+\Delta t) = 1 \mid I_i(t) = 0)\, P(I_i(t) = 0)\Big)$$

$$= \sum_{i=1}^{n}(1 - \alpha\Delta t)P(I_i(t) = 1) + \beta\Delta t \sum_{j=1}^{n} A_{ij} I_j(t) P(I_i = 0$$

$$= [I](t) - \alpha\Delta t[I](t) + \sum_{i=1}^{n}\sum_{j=1}^{n} \beta\Delta t\, A_{ij}\, I_j(t) P(I_i = 0)$$

$$\Rightarrow [I](t+\Delta t) = [I](t) - \alpha \Delta t [I](t) + \beta \Delta t [IS](t)$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{[I](t+\Delta t) - [I](t)}{\Delta t} = -\alpha [I](t) + \beta [IS](t).$$

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* This is not a closed system of equations!!
Need equations for [SS], [IS], [II] to close system.

## Moment Closure

- Assume a network in which each node has the same degree K.
- There are [I] infected nodes
- $[I]/n$ = proportion of infected population.
- On average a susceptible node has $\frac{\langle k \rangle [I]}{n}$ infected neighbors.

$$\Rightarrow [SI] \approx \frac{\langle k \rangle}{n} [I][S].$$

Approximate closed equations:

$$\boxed{\begin{aligned} [\dot{S}] &\approx -\frac{\beta \langle k \rangle}{n} [I] \cdot [S] + \alpha [I] \\ [\dot{I}] &\approx \frac{\beta \langle k \rangle}{n} [I][S] - \alpha [I]. \end{aligned}}$$

Covariance and why not so bad.

$$[\dot{S}] = -\beta[IS] + \alpha[I]$$
$$[\dot{I}] = \beta[IS] - \alpha[I]$$

Recall:

$$\operatorname{cov}(X,Y) = \mathbb{E}((X-\mathbb{E}[X])(Y-\mathbb{E}[Y]))$$
$$= \mathbb{E}(X \cdot Y) - \mathbb{E}[X] \cdot \mathbb{E}[Y]$$
$$\Rightarrow \mathbb{E}[X \cdot Y] = \operatorname{cov}(X,Y) + \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

$$\Rightarrow [\dot{S}] = -\frac{\beta\langle k\rangle}{n}[I]\cdot[S] + \alpha[I] + \gamma \operatorname{cov}(I,S)$$
$$[\dot{I}] = \frac{\beta\langle k\rangle}{n}[I]\cdot[S] - \alpha[I] - \gamma \operatorname{cov}(I,S)$$

The covariance can be shown to be positive

$\Rightarrow$ O.D.E. Model overestimates impact of disease.