Lecture 7: Edge Dynamics.

What if we want to modify edge dynamics??

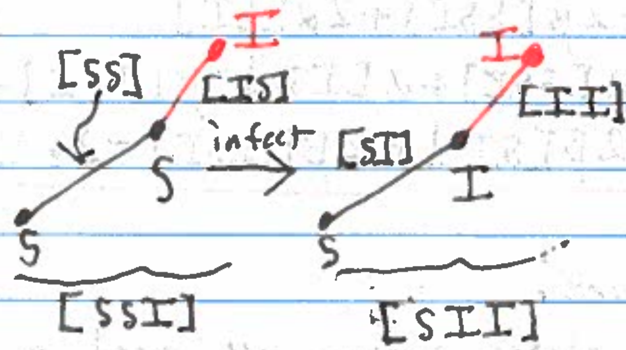
$[\dot{S}] = -\beta[IS] + \alpha[I]$

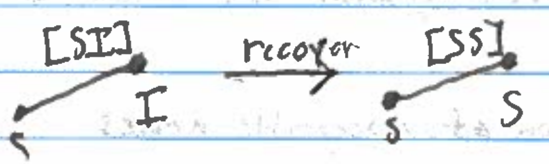
$[\dot{I}] = \beta[IS] - \alpha[I]$

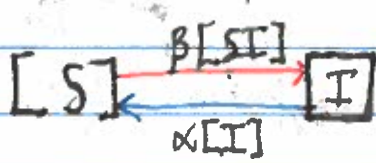
$[\dot{SS}] = ?$

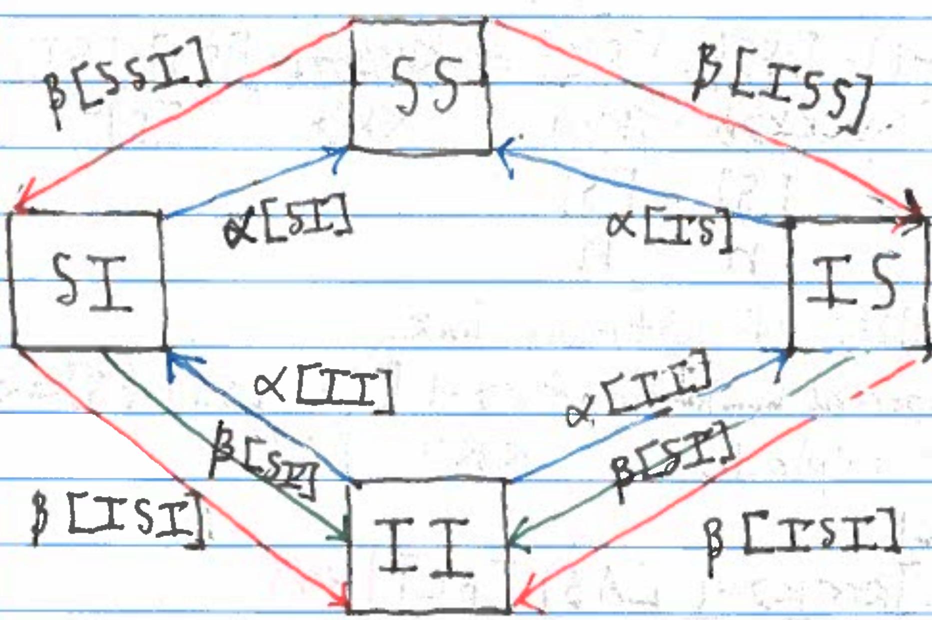
$[\dot{SI}] = ?$

$[\dot{II}] = ?$









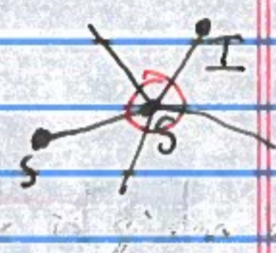
$$[\dot{SS}] = -\beta[SSI] - \beta[ISS] + \alpha[SI] + \alpha[IS]$$
$$[\dot{SI}] = \beta[SSI] + \alpha[II] - \beta[ISI] - \alpha[SI]$$
$$[\dot{IS}] = \beta[ISS] + \alpha[II] - \beta[ISI] - \alpha[IS]$$
$$[\dot{II}] = \beta[SI] + \beta[IS] + 2\beta[ISI] - 2\alpha[II]$$

However, $[SI] = [IS]$ and $[SSI] = [ISS]$

$$\Rightarrow \boxed{\begin{aligned} [\dot{SS}] &= -2\beta[SSI] + 2\alpha[SI] \\ [\dot{SI}] &= \beta[SSI] + \alpha[II] - \beta[ISI] - \alpha[SI] - \beta[IS] \\ [\dot{II}] &= 2\beta[SI] + 2\beta[ISI] - 2\alpha[II] \end{aligned}}$$

<u>Moment Closure:</u>

To create a closed system we need a rule for approximating triple links.



- $\frac{[S]}{n}$ proportion of susceptible nodes
- $\frac{[SS]}{\langle k\rangle n}$, $\frac{[SI]}{\langle k\rangle n}$ proportion of edges

- $$\langle k\rangle(\langle k\rangle - 1) \cdot \frac{\frac{[SS]}{\langle k\rangle n} \cdot \frac{[SI]}{\langle k\rangle n}}{\frac{[S]}{n} \cdot \frac{[S]}{n}} = \frac{\langle k\rangle - 1}{\langle k\rangle} \cdot \frac{[SS] \cdot [SI]}{[S]^2}$$

  = probability of a triple link.

$\Rightarrow [SSI]$ = expected number of triple links $= \frac{\langle k\rangle - 1}{\langle k\rangle} \frac{[SS] \cdot [SI]}{[S]^2} [S] = \frac{\langle k\rangle - 1}{\langle k\rangle} \frac{[SS][SI]}{[S]}$

- In general

$$\boxed{[ABC] \approx \frac{\langle k\rangle - 1}{\langle k\rangle} \frac{[AB] \cdot [BC]}{[B]^2}} \quad (*)$$

Theorem - Given the moment closure (*)

$$[SS]+[SI]=\langle k\rangle[S]$$
$$[SI]+[II]=\langle k\rangle[I]$$

proof:

Let

$$A=[SS]+[SI]-\langle k\rangle[S]$$
$$B=[SI]+[II]-\langle k\rangle[I]$$

$$\Rightarrow \dot{A}=[\dot{S}S]+[\dot{S}I]-\langle k\rangle[\dot{S}]$$
$$=-\beta([SSI]+[ISI]+[IS]-\langle k\rangle[IS])$$
$$+\alpha([SI]+[II]-k[\dot{I}])$$
$$=-\beta\left(\frac{\langle k\rangle-1}{\langle k\rangle}\frac{[SS][SI]}{[S]}+\frac{\langle k\rangle-1}{\langle k\rangle}\frac{[SI]^2}{[S]}-(\langle k\rangle-1)[SI]\right)$$
$$+\alpha B$$
$$=-\beta\frac{\langle k\rangle-1}{\langle k\rangle}\frac{[SI]}{[S]}([SS]+[SI]-\langle k\rangle[S])+\alpha B$$
$$=-\beta\frac{\langle k\rangle-1}{\langle k\rangle}\frac{[SI]}{[S]}A+\alpha B$$

A similar calculation yields

$$\dot{A}=-\frac{\beta(\langle k\rangle-1)}{\langle k\rangle}\frac{[SI]}{[S]}A+\alpha B$$
$$\dot{B}=\alpha A-\frac{\beta(\langle k\rangle-1)}{\langle k\rangle}\frac{[SI]}{[S]}B.$$

$A=B=0$ is a fixed point, thus if intial conditions satisfy $A=B=0$ then A, B are conserved. ∎

We have three independent conservation laws:

$$[S]+[I]=n$$
$$[SS]+2[SI]+[II]=\langle k\rangle n$$
$$[SS]+[SI]=\langle k\rangle[S]$$

With these moment closures and conservation laws we need only consider $[S]$ and $[SS]$.

$$[\dot{S}]=-\beta[IS]+\alpha[I]$$
$$[\dot{SS}]=-2\beta[SSI]+2\alpha[SI]$$

$$\Rightarrow [\dot{S}]=-\beta(\langle k\rangle[S]-[SS])+\alpha(n-[S])$$
$$[\dot{SS}]=-2\beta\frac{(\langle k\rangle-1)}{\langle k\rangle}\frac{[SS][SI]}{[S]^2}+2\alpha(\langle k\rangle[S]-[SS])$$

$$\boxed{\begin{aligned}\Rightarrow [\dot{S}]&=-(\beta\langle k\rangle+\alpha)[S]+\beta[SS]+\alpha n.\\ [\dot{SS}]&=-2\beta\frac{(\langle k\rangle-1)}{\langle k\rangle}\frac{[SS](\langle k\rangle[S]-[SS])}{[S]^2}+2\alpha(\langle k\rangle[S]-[SS])\end{aligned}}$$