Lecture 9: Building Complex Epidemiological Models.

SEIR Model:

$$\dot{S} = \Lambda - \beta S I - \nu S \qquad \Lambda - \text{birth rate}$$
$$\dot{E} = \beta S I - \gamma E - \nu E \qquad \beta - \text{force of infection}$$
$$\dot{I} = \gamma E - \alpha I - \nu I \qquad \gamma - \text{rate of symptom formation}$$
$$\dot{R} = \alpha I - \nu R \qquad \alpha - \text{recovery rate}$$
$$\nu - \text{death rate.}$$

↑

Not conservative.

Disease free equilibrium?

$$S = \Lambda/\nu, \quad I = E = R = 0.$$

$$J = \begin{bmatrix} -\beta I - \nu & 0 & -\beta S & 0 \\ \beta I & -\gamma - \nu & \beta S & 0 \\ 0 & \gamma & -\alpha - \nu & 0 \\ 0 & 0 & \alpha & -\nu \end{bmatrix}$$

$$\Rightarrow J(\Lambda/\nu, 0, 0, 0) = \begin{bmatrix} -\nu & 0 & -\beta \Lambda/\nu & 0 \\ 0 & -\gamma - \nu & \beta \Lambda/\nu & 0 \\ 0 & \gamma & -\alpha - \nu & 0 \\ 0 & 0 & \alpha & -\nu \end{bmatrix}$$

$$\lambda_1, \lambda_2 = -\nu$$

Look at submatrix:

$$A = \begin{bmatrix} -\gamma - \nu & \beta \Lambda/\nu \\ \gamma & -\alpha - \nu \end{bmatrix}$$

$Tr(A) = -\eta - 2\nu - \alpha$

$Det(A) = (\eta+\nu)(\alpha+\nu) - \beta\, \Lambda/\nu\, \eta$

If the disease free state is stable the $Det(A) > 0$.

$\Rightarrow (\eta+\nu)(\alpha+\nu) > \dfrac{\beta \Lambda \eta}{\nu}$

$\Rightarrow \dfrac{\beta \Lambda \eta}{\nu(\eta+\nu)(\alpha+\nu)} < 1$

$\Rightarrow \boxed{R_0 = \dfrac{\beta \Lambda \eta}{\nu(\eta+\nu)(\alpha+\nu)}}$

Asymptomatic Stage:

$\dot{S} = \Lambda - \beta S(I+qA) - \nu S$

$\dot{E} = \beta S(I+qA) - (\eta+\nu)E$

$\dot{I} = p\eta E - (\alpha+\nu)I$

$\dot{A} = (1-p)\eta E - (\gamma+\nu)A \rightarrow$ Asymptomatic

$\dot{R} = \alpha I + \gamma A - \nu R$

New parameters

- $q\beta$ reduced transmission rate.
- η rate of becoming infectious.
- p probality of becoming symptomatic

$(\Lambda/\nu, 0, 0, 0, 0)$ = disease free equilibrium.

$$J = \begin{bmatrix} -\beta(I+qA)-\nu & 0 & -\beta S & -\beta q S & 0 \\ \beta(I+qA) & -(\eta+\nu) & \beta S & \beta q S & 0 \\ 0 & p\eta & -(\alpha+\nu) & 0 & 0 \\ 0 & (1-p)\eta & 0 & -(\gamma+\nu) & 0 \\ 0 & 0 & \alpha & \gamma & -\nu \end{bmatrix}$$

$$J(\Lambda/\nu, 0,0,0,0) = \begin{bmatrix} -\nu & 0 & -\beta \Lambda/\nu & -\beta q \Lambda/\nu & 0 \\ 0 & -(\eta+\nu) & \beta \Lambda/\nu & \beta q \Lambda/\nu & 0 \\ 0 & p\eta & -(\alpha+\nu) & 0 & 0 \\ 0 & (1-p)\eta & 0 & -(\gamma+\nu) & 0 \\ 0 & 0 & \alpha & \gamma & -\nu \end{bmatrix}$$

$\lambda_1, \lambda_2 = -\nu.$

Define

$$A = \begin{bmatrix} -(\eta+\nu) & \beta \Lambda/\nu & \beta q \Lambda/\nu \\ p\eta & -(\alpha+\nu) & 0 \\ (1-p)\eta & 0 & -(\gamma+\nu) \end{bmatrix}$$

Need to find eigenvalues of above. Actually just need all eigenvalues to be negative.

Theorem - Routh-Hurwitz Criteria - Consider the n-th degree polynomial with real coefficients.

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_{n-1}\lambda + a_n.$$

Define n Hurwitz matrices

$$H_1 = [a_1], \quad H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}, \quad H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}$$

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

All roots of the polynomial have negative real part If and only if

$$\mathrm{Det}\, H_j > 0, \quad j = 1, \dots, n.$$

Example:

$$\lambda^2 - \mathrm{Tr}(J)\lambda + \det(J).$$

$$H_1 = [-\mathrm{Tr}(J)] \quad \Rightarrow \det(H_1) = -\mathrm{Tr}(J)$$

$$H_2 = \begin{bmatrix} -\mathrm{Tr}(J) & 1 \\ 0 & \mathrm{Det}(J) \end{bmatrix} \quad \Rightarrow \det(H_2) = -\mathrm{Tr}(J)\,\mathrm{Det}(J).$$

Therefore, roots are negative if

$$\mathrm{Tr}(J) < 0, \quad \mathrm{Det}(J) > 0.$$

Return to problem:

We need to derive the characteristic polynomial for A:

$$p(\lambda) = \det\left(\begin{bmatrix} -(\eta+\omega+\lambda) & \beta\Lambda/\omega & \beta q\Lambda/\omega \\ p\eta & -(\alpha+\omega+\lambda) & 0 \\ (1-p)\eta & 0 & -(\gamma+\omega+\lambda) \end{bmatrix}\right)$$

Lets find a rule for cubic polynomials

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

$$H_1 = [a_1],\ H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix},\ H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{bmatrix}$$

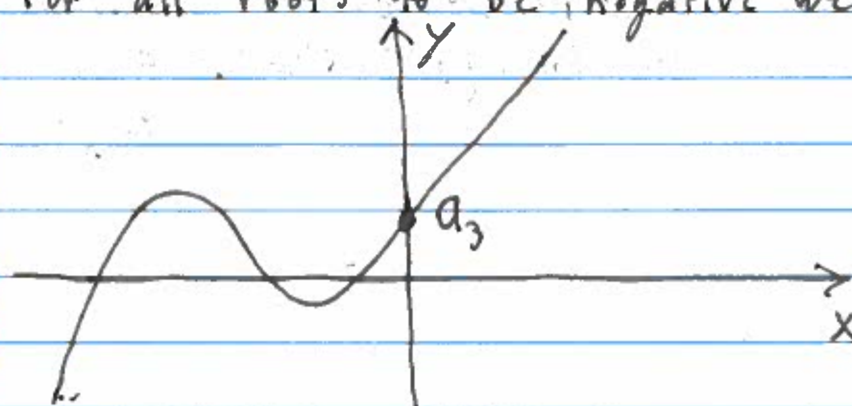
Routh-Hurwitz

$\Rightarrow a_1 > 0 \qquad \Rightarrow a_1 > 0$

$a_1 a_2 - a_3 > 0 \Rightarrow a_3 < a_1 a_2$

$a_1 a_2 a_3 - a_3^2 > 0 \Rightarrow a_3(a_1 a_2 - a_3) > 0$

For all roots to be negative we need $a_3 > 0$



Is this sufficient? If so:

$$a_3 = (\alpha+\omega)(\eta+\omega)(\gamma+\omega) - (\omega+\alpha)(1-p)\eta q\beta\Lambda/\omega - (\gamma+\omega)p\eta\beta\Lambda/\omega$$

$$\Rightarrow \boxed{R_0 = \frac{(1-p)\eta q\beta\Lambda/\omega}{(\eta+\omega)(\gamma+\omega)} + \frac{p\eta\beta\Lambda/\omega}{(\alpha+\omega)(\eta+\omega)}}$$