Lecture 1: Algebra of Complex Numbers

History:

- Solving linear equations
  $$ax+b=0$$
  $$\Rightarrow x=-b/a$$
  The rational numbers are sufficient.

- Solving quadratic equations
  $$x^2=2$$
  $$\Rightarrow x=\sqrt{2}$$
  This requires irrational numbers.

- Solving cubics and $x^2=-1$ requires a new number.

Definition: $\sqrt{-1}=i$

Examples:

- $i^2=-1$
- $i^3=i^2\cdot i=-i$
- $i^4=i^3\cdot i=1$
- $i^5=i$
- $\vdots$

i is associated with periodic patterns and rotations.

Definition: A complex number $z\in\mathbb{C}$ is a number in the form

$$z=x+iy, \rightsquigarrow \text{standard form.}$$

where $x,y\in\mathbb{R}$.

- $x=\mathrm{Re}(z)$ is called the real part of z.
- $y=\mathrm{Im}(z)$ is called the imaginary part of z.

$\mathbb{C}$ = set of all complex numbers.

Examples:

1. $i^{-1} = -i$

~~proof~~:

$$i^{-1} = \frac{1}{i}$$

$$\Rightarrow \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{-1} = -i.$$

2. $i^{-2} = (i^{-1})^2 = (-i)^2 = (-1)^2 i^2 = -1.$

3. Show that for all $z \in \mathbb{C}$, $Re(iz) = -Im(z)$.

~~proof~~:

Let $z = x + iy = Re(z) + i\,Im(z)$. Therefore,

$$\begin{aligned} Re(iz) &= Re(i(Re(z) + i\,Im(z))) \\ &= Re(ix - y) \\ &= -y \\ &= -Im(z). \end{aligned}$$

4. Let $z \in \mathbb{C}$ such that $Re(z) > 0$. Prove that $Re(1/z) > 0$.

~~proof~~:

Let $z = x + iy \in \mathbb{C}$ where $x, y \in \mathbb{R}$. Therefore,

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

and thus

$$Re(1/z) = x/x^2+y^2 = Re(z)/x^2+y^2 > 0.$$

5. What is $\sqrt{i}$?

Solution:

$$\sqrt{i} = x + iy$$

$$\Rightarrow i = x^2 - y^2 + 2ixy$$

Consequently, comparing coefficients, it follows that

$$x^2 - y^2 = 0 \quad \text{and} \quad 2xy = 1$$

$$\Rightarrow y = 1/2x \Rightarrow x^2 - \frac{1}{4x^2} = 0 \Rightarrow 4x^4 = 1 \Rightarrow x = 1/\sqrt{2},\ y = 1/\sqrt{2}.$$

Therefore,

$$\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}. \rightsquigarrow \text{Principal root}$$

6. Let $z = 2+3i$, $w = -1+2i$.

$$z \cdot w = (2+3i)(-1+2i)$$
$$= -2 + i - 6$$
$$= -8 + i$$

$$\frac{z}{w} = \frac{2+3i}{-1+2i} = \frac{2+3i}{-1+2i} \cdot \frac{-1-2i}{-1-2i} = \frac{-2-5i+6}{5} = \frac{4}{5} - i.$$