Lecture #11: Harmonic Functions

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{(Laplace's Equation).}$$

Solution to Laplace's equation are called harmonic.

Observation:

If $f: \mathbb{C} \to \mathbb{C}$ is analytic and $f(x,y) = u(x,y) + i\,v(x,y)$ then u, v are harmonic.

proof:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y} = -\frac{\partial^2 v}{\partial x \partial y}, \qquad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}, \quad \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y},$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Example:

Does there exist an analytic function whose real part is given by

$$u(x,y) = x^3 - 3xy^2 + y$$

and if so, find the function.

$$\frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial^2 u}{\partial y^2} = -6x$$

$\Rightarrow u$ is harmonic. The harmonic conjugate v satisfies

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow 3x^2 - 3y^2 = \frac{\partial v}{\partial y}$$

$$\Rightarrow v = 3x^2y - y^3 + \Psi(x)$$

$$\Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow 6xy + \psi'(x) = 6xy - 1$$

$$\Rightarrow \psi'(x) = -1$$

$$\Rightarrow \psi(x) = -x$$

Therefore,

$$f(x,y) = x^3 - 3xy^2 + y + i(3x^2y - y^3 - x)$$