Lecture 12: Polynomials and Rational Functions

Polynomials:

$$p_n(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n, \quad a_i, z \in \mathbb{C}$$

Example:

$$p(z) = -2z^3 - 4z^2 + 10z + 12$$

- 3 roots in $\mathbb{C}$ by fundamental theorem of algebra.
- $p_3(-1) = 0$
- Factor:

$$\begin{array}{r|l}
 & -2z^2 - 2z + 12 \\ \hline
z+1 & -2z^3 - 4z^2 + 10z + 12 \\
 & \underline{-2z^3 - 2z^2} \\
 & \quad -2z^2 + 10z \\
 & \quad \underline{-2z^2 - 2z} \\
 & \qquad 12z + 12 \\
 & \qquad \underline{12z + 12} \\
 & \qquad\quad 0
\end{array}$$

$\Rightarrow -2(z^2 + z - 6)(z+1) = p(z)$

$\Rightarrow -2(z+3)(z-2)(z+1) = p(z)$

- Express $p_3(z)$ in powers of $z-1$:

$$p_3(z) = 12 + 10z - 4z^2 - 2z^3 = c_0 + c_1(z-1) + c_2(z-1)^2 + c_3(z-1)^3$$

- $p_3(1) = 12 - 4 - 2 + 10$

  $\Rightarrow c_0 = 10$
- $p_3'(z) = 10 - 8z - 6z^2 = c_1 + 2c_2(z-1) + 3c_3(z-1)^2$

  $p_3'(1) = 10 - 8 - 6 = c_1$

  $\Rightarrow c_1 = -4$
- $p_3''(z) = -8 - 12z = 2c_2 + 6c_3(z-1)$

  $p_3''(1) = -8 - 12 = 2c_2$

  $\Rightarrow c_2 = -10$
- $p_3'''(z) = -12 = 6c_3$

  $\Rightarrow c_3 = -2$

$\Rightarrow p_3(z) = 10 - 4(z-1) - 10(z-1)^2 - 2(z-1)^3.$

Rational Functions:

$$R(z) = a\,\frac{(z-z_1)\cdots(z-z_n)}{(z-\xi_1)\cdots(z-\xi_m)} = a\,\frac{p(z)}{q(z)} \quad \left(\begin{array}{l}\text{common terms}\\ \text{cancelled}\end{array}\right)$$

- roots satisfy $p(z)=0$
- poles satisfy $q(z)=0$
- order of a pole is the order of the root to $q(z)$.

Example:

$$R(z) = \frac{(3z+3i)(z^2-4)}{(z-2)(z^2+1)^2}$$

- Simplify:

$$R(z) = \frac{3(z+i)\cancel{(z-2)}(z+2)}{\cancel{(z-2)}(z+i)^2(z-i)^2}$$

$$= \frac{3(z+i)(z+2)}{(z+i)^2(z-i)^2}$$

$z=\pm i$ are poles of order 2

- Partial Fractions:

$$\frac{3(z+i)(z+2)}{(z+i)^2(z-i)^2} = \frac{3(z^2+2iz+2z+2i)}{(z+i)^2(z-i)^2} = \frac{A}{(z+i)^2} + \frac{B}{(z-i)^2} + \frac{C}{z+i} + \frac{D}{z-i}$$

$$\Rightarrow 3(z^2+2iz+2z+2i) = A(z-i) + B(z+i) + C(z+i)(z-i)^2 + D(z-i)(z+i$$

$$\Rightarrow 3(z^2+2iz+2z+2i) = A(z-i) + B(z+i) + C(z+i)(z^2-2iz-1) + D(z-i)(z^2+2z$$

$$\Rightarrow 3(z^2+2iz+2z+2i) = A(z-i) + B(z+i) + C(z^3-2iz^2-z+iz^2+2z-i) + D(z^3+2iz^2-z-iz^2+2z+i)$$

$$\Rightarrow 3(z^2+2iz+2z+2i) = (-A+B-C+D)i + (A+B+C+D)z + (-C+D)iz^2 + (C+D)z^3$$

$$\Rightarrow \begin{aligned} -A+B-C+D &= 6 \\ A+B+C+D &= 6i+6 \\ -C+D &= 6i \\ C+D &= 0 \end{aligned}$$

$\Rightarrow C = -D$

$\Rightarrow D = 3i$ and $C = -3i$

Therefore,

$$-A + B + 3i + 3i = 6$$

$$A + B = 6i + 6$$

$\Rightarrow 2B + 6i = 6i + 12$

$\Rightarrow B = 6$ and $A = 6i$

Therefore,

$$\frac{3(z+i)(z+2)}{(z+i)^2(z-i)^2} = \frac{6i}{(z+i)^2} + \frac{6}{(z-i)^2} \frac{-3i}{z+i} + \frac{3i}{z-i}.$$