Lecture 13: Transcendental Functions

Exponential Function

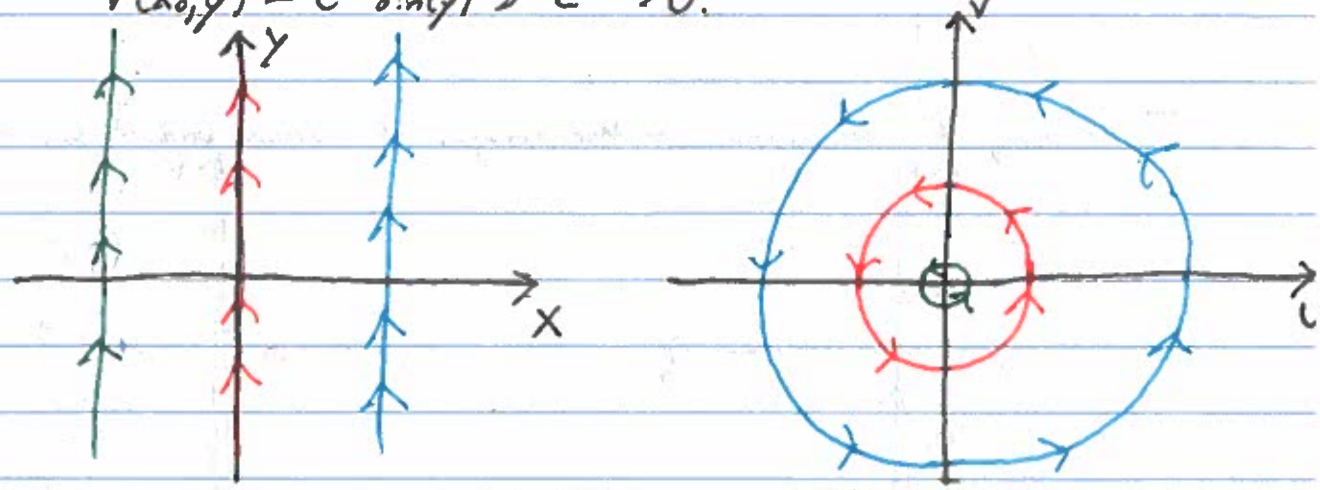
$e^z = e^{x+iy} = e^x e^{iy+2n\pi i}$ → periodic in the complex plane.

$= e^x(\cos(y) + i\sin(y))$

$= e^x\cos(y) + ie^x\sin(y)$

For $x = x_0$ fixed we have

$$\left.\begin{aligned} u(x_0, y) &= e^{x_0}\cos(y) \\ v(x_0, y) &= e^{x_0}\sin(y) \end{aligned}\right\}$$ parametrization of circle of radius $e^{x_0} > 0$.



Domain of $f(z) = e^z$ is $\mathbb{C}$, range is $\mathbb{C}\setminus\{0\}$.

Trig Functions

1. $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$

$\Rightarrow \sin(z) = \dfrac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2i}$

$= (e^{-y}(\cos(x) + i\sin(x)) - e^{y}(\cos(x) + i\sin(x)))/2i$

$= \dfrac{\cos(x)(e^{-y} - e^{y}) + i\sin(x)(e^{y} + e^{-y})}{2i}$

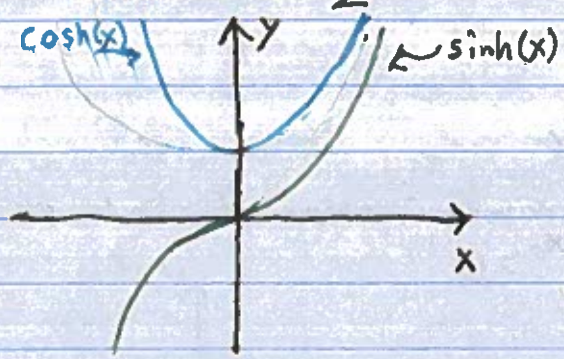
$= \dfrac{-\cos(x)\sinh(y) + i\sin(x)\cosh(y)}{i}$

$= \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

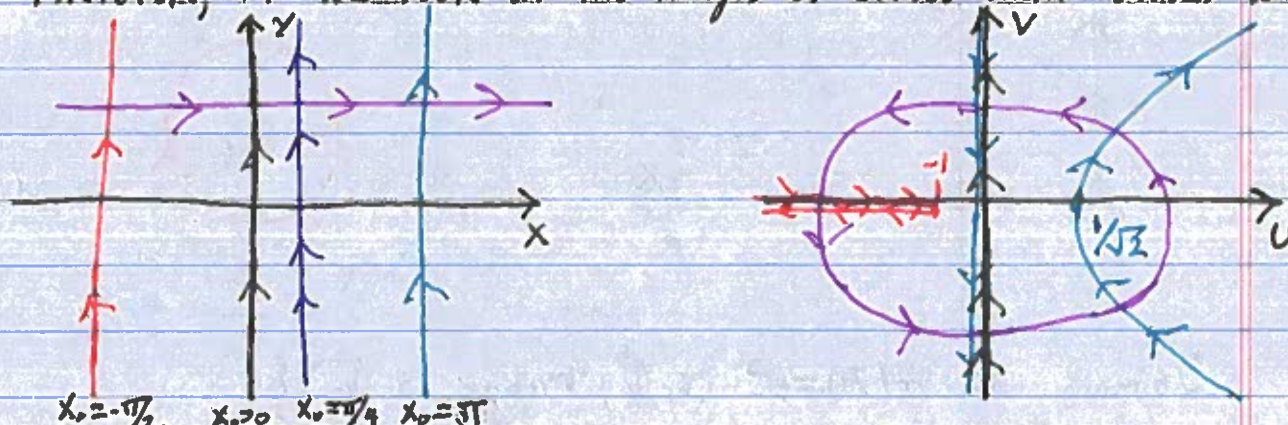
__Recall:__

For real x:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$



Therefore, if we look at the images of curves under $\sin(z)$ we have:



$u(x_0, y) = \sin(x_0)\cosh(y)$

$v(x_0, y) = \cos(x_0)\sinh(y)$

a.) $x_0 = 0 \Rightarrow u(x_0, y) = 0$

$\quad v(x_0, y) = \sinh(y)$

b.) $x_0 = \pi \Rightarrow u(x_0, y) = 0$

$\quad v(x_0, y) = -\sinh(y)$

c.) $x_0 = -\pi/2 \Rightarrow u(x_0, y) = -\cosh(y)$

$\quad v(x_0, y) = 0$

d.) $x_0 = \pi/4 \Rightarrow u(x_0, y) = \frac{1}{\sqrt{2}}\cosh(y)$

$\quad v(x_0, y) = \frac{1}{\sqrt{2}}\sinh(y)$

e.) $y_0 \Rightarrow u(x, y_0) = \sin(x)\cosh(y_0)$

$\quad v(x, y_0) = \cos(x)\sinh(y_0)$

2. $\cos(z) = \dfrac{e^{iz}+e^{-iz}}{2}$

$$= \frac{e^{ix-y}+e^{-ix+y}}{2}$$

$$= \frac{e^{-y}(\cos(x)+i\sin(x)) + e^{y}(\cos(x)-i\sin(x))}{2}$$

$$= \frac{\cos(x)(e^{y}+e^{-y}) - i\sin(x)(e^{y}-e^{-y})}{2}$$

$$= \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

$$= u(x,y) + i\,v(x,y)$$

$\Rightarrow \dfrac{\partial u}{\partial x} = -\sin(x)\cosh(y)$, $\quad \dfrac{\partial v}{\partial x} = -\cos(x)\sinh(y)$

$\dfrac{\partial u}{\partial y} = \cos(x)\sinh(y)$, $\quad \dfrac{\partial v}{\partial y} = -\sin(x)\cosh(y)$

Cauchy-Riemann equations are satisfied so:

$$\frac{d}{dz}\cos(z) = -\sin(x)\cosh(y) - i\cos(x)\sinh(y)$$

$$= -\sin(z).$$

A similar calculation shows that

$$\frac{d}{dz}\sin(z) = \cos(z).$$

3. $\cosh(z) = \dfrac{e^{z}+e^{-z}}{2} \Rightarrow \cosh(iz) = \cos(z)$

4. $\sinh(z) = \dfrac{e^{z}-e^{-z}}{2} \Rightarrow \sinh(iz) = i\sin(z)$