Lecture 14: Logarithmic Functions

Motivation-

$$z=|z|e^{i\theta+2in\pi}, \quad \theta=\mathrm{Arg}(z)\in(-\pi,\pi]$$

$$\Rightarrow \log(z)=\log(|z|e^{i\theta+2in\pi})$$
$$=\log(|z|)+\log(e^{i\theta+2in\pi})$$
$$=\mathrm{Log}(|z|)+i\theta+2in\pi$$
$$=\mathrm{Log}(|z|)+i\arg(z)+2in\pi$$

$$\mathrm{Log}(z)=\mathrm{Log}(|z|)+i\,\mathrm{Arg}(z)$$

Branch cuts and branches -

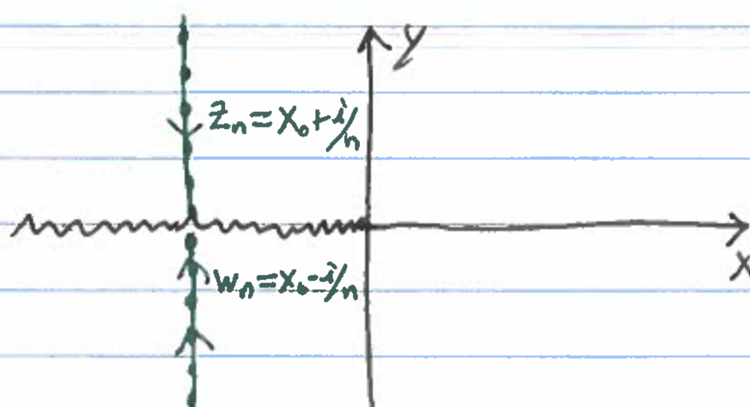
- $\mathrm{Log}(z)$ is not continuous along the line $\mathrm{Im}(z)=0$, $\mathrm{Re}(z)\leq 0$

To see why, let $z_0=x_0$, where $x_0<0$. Consider the two sequences

$$z_n=x_0+i/n$$
$$w_n=x_0-i/n$$

$$\Rightarrow \mathrm{Log}(z_n)=\mathrm{Log}((x_0^2+1/n^2)^{1/2})+i\,\mathrm{Arg}(x_0+i/n)$$
$$\mathrm{Log}(w_n)=\mathrm{Log}((x_0^2+1/n^2)^{1/2})+i\,\mathrm{Arg}(x_0-i/n)$$
$$\Rightarrow \lim_{n\to\infty}\mathrm{Log}(z_n)=\mathrm{Log}(|x_0|)+i\pi$$
$$\lim_{n\to\infty}\mathrm{Log}(w_n)=\mathrm{Log}(|x_0|)-i\pi$$

The line along which Log is discontinuous is called a branch cut:



Other branch cuts can be made depending on the choice $[\tau, \tau+2\pi]$.

Differentiability

$$f(z) = \mathrm{Log}(\sqrt{x^2+y^2}) + i\tan^{-1}(y/x)$$
$$= u(x,y) + iv(x,y)$$

- $\dfrac{\partial u}{\partial x} = \dfrac{1}{(x^2+y^2)^{1/2}} \dfrac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \dfrac{x}{x^2+y^2}$ $\qquad \dfrac{\partial v}{\partial x} = \dfrac{1}{1+y^2/x^2} \dfrac{-y}{x^2} = \dfrac{-y}{x^2+y^2}$

- $\dfrac{\partial u}{\partial y} = \dfrac{y}{x^2+y^2}$ $\qquad \dfrac{\partial v}{\partial y} = \dfrac{1}{1+y^2/x^2} x = \dfrac{x}{x^2+y^2}$

Therefore, $f'(z) = \dfrac{x}{x^2+y^2} \dfrac{-iy}{x^2+y^2} = \dfrac{1}{z}$.

However, f is not differentiable along the negative real axis.