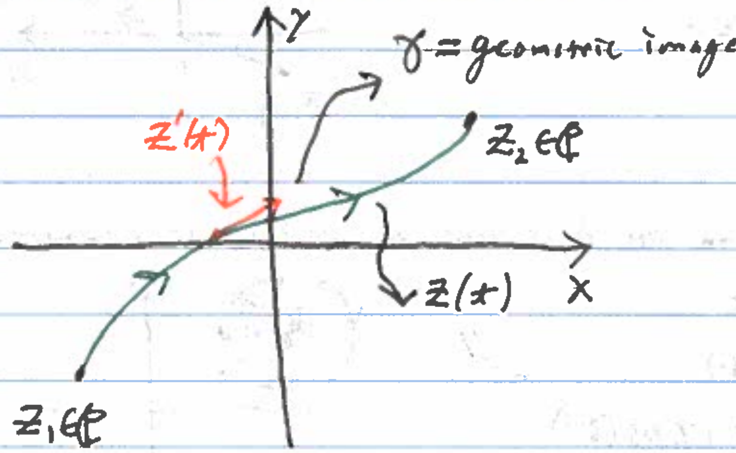
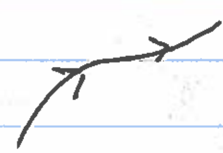
Lecture 16: Contours

Parametrization of Curves in $\mathbb{C}$



* $z(t) : [a,b] \to \mathbb{C}$ is a smooth parametrization of γ if
  1. $z(a) = z_1$ and $z(b) = z_2$
  2. z' is continuous and $z' \neq 0$
  3. z is one-to-one
* γ is a smooth curve if it is the image of a smooth parametrization.

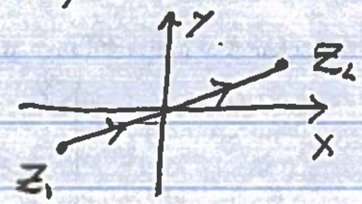
Examples:

- 
  not smooth
- 
  not smooth
- 
  not smooth
- 
  Smooth
- 
  Smooth

Examples

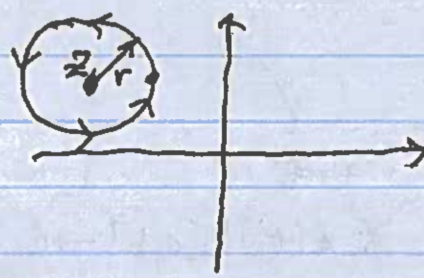
1. Find a parametrization of the line connecting $z_1, z_2 \in \mathbb{C}$. For $t \in [0,1]$ define

$$z(t) = (1-t)z_1 + t z_2$$
$$z'(t) = -z_1 + z_2$$



2. Find a paramerrization of a circle of radius r centered at $z_1 \in \mathbb{C}$.
For $t \in [0, 2\pi]$ define

$$z(t) = x(t) + iy(t)$$
$$x(t) = Re(z_1) + r\cos(t)$$
$$y(t) = Im(z_1) + r\sin(t)$$

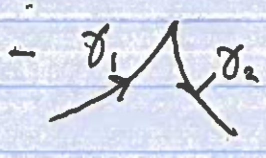


$$\Rightarrow z(t) = (Re(z_1) + r\cos(t)) + i(Im(z_1) + r\sin(t))$$
$$= z_1 + re^{it}$$
$$\Rightarrow z_1'(t) = rie^{it}$$

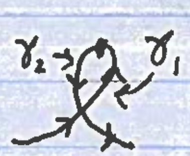
Contours

A contour Γ is either a single point or the union of a sequence of parametrized curves $(\gamma_1, \dots, \gamma_n)$ such that γ_k and γ_{k+1} share terminal and initial points respectively.

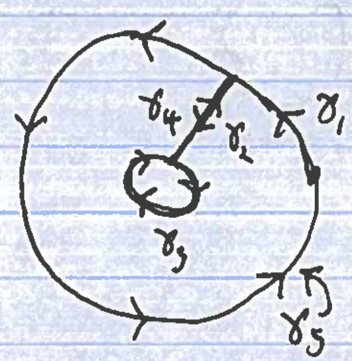
Examples

- 

$\Gamma = \gamma_1 \cup \gamma_2$

- 

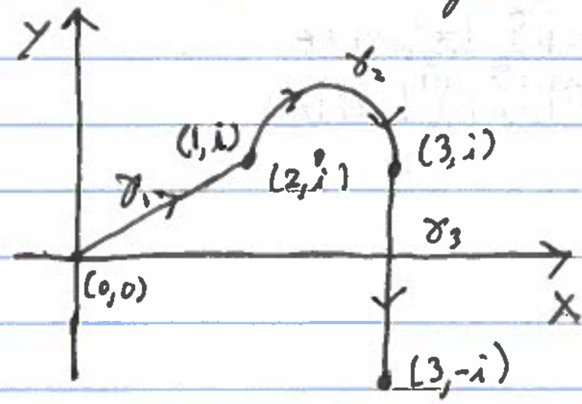
$\Gamma = \gamma_1 \cup \gamma_2$

- 

$\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup$

Example:

Parametrize the following contour



γ_1: $z_1 : [0,1] \to \mathbb{C}$ defined by

$$z_1(t) = (1-t)\cdot 0 + t(1+i) = t(1+i)$$
$$z_1'(t) = 1+i$$

γ_2: $z_2 : [0,\pi] \to \mathbb{C}$ defined by

$$z_2(t) = (2+i) + e^{-i(t+\pi)}.$$
$$z_2'(t) = -i e^{-i(t+\pi)}$$

γ_3: $z_3 : [0,1] \to \mathbb{C}$ defined by

$$z_3(t) = (1-t)(3+i) + t(3-i)$$
$$z_3'(t) = 3-i-3-i = -2i$$

$\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ is a contour.

Length of Γ:

$$L[\gamma] = \int_\Gamma ds, \quad ds = \sqrt{dx^2+dy^2}$$

However,

$$\begin{aligned} ds &= \sqrt{dx^2+dy^2} \\ &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \sqrt{|z'(t)|^2}\, dt \\ &= |z'(t)|\, dt \end{aligned}$$

Therefore,

$$\int_{\gamma} ds = \int_{\gamma_1} ds + \int_{\gamma_2} ds + \int_{\gamma_3} ds$$
$$= \int_{\gamma_1} |z_1'(t)|dt + \int_{\gamma_2} |z_2'(t)|dt + \int_{\gamma_3} |z_3'(t)|dt$$
$$= \int_0^1 |1+i|dt + \int_0^{\pi} |-ie^{-i(t+\pi)}|dt + \int_0^1 |-2i|dt$$
$$= \sqrt{2} + \pi + 2.$$