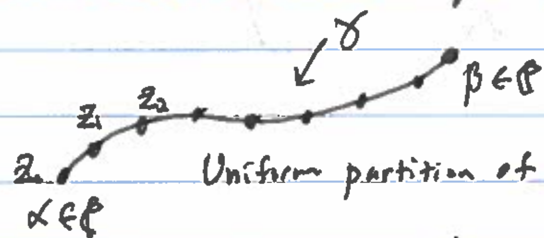
Lecture 17: Contour Integrals



Uniform partition of width $\Delta z = \Delta x + i\Delta y$

$$\int_\gamma \underbrace{f(z)}dz = \lim_{N\to\infty} \sum_{i=1}^{N} f(z_i)\Delta z = \lim_{N\to\infty} \sum_{i=1}^{N} f(z_i)\frac{\Delta z}{\Delta t}\Delta t$$

Assume f is continuous

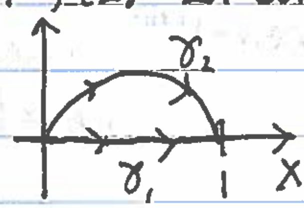
Let $z(t)$ be a parametrization of γ satisfying:

$z(a)=\alpha,\ z(b)=\beta$

$$\Rightarrow \int_\gamma f(z)dz = \int_a^b \underbrace{f(z(t))z'(t)dt}_{\text{regular integration}} \quad (dz \text{ "=" } z'(t)dt).$$

Example:

Let $f(z)=z$. Compute $\int_\gamma f(z)dz$ over the following paths.



a) $z_1(t) = (1-t)\,0 + 1\cdot t,\ z_1'(t) = 1$

$$\Rightarrow \int_{\gamma_1} f(z)dz = \int_0^1 z_1(t)\cdot z_1'(t)dt = \int_0^1 t\,dt = 1/2.$$

(b) $z_2(t) = 1/2 + 1/2\, e^{i(\pi - t)},\ z_2'(t) = -1/2\, i e^{i(\pi-t)}$

$$\Rightarrow \int_{\gamma_2} f(z)\,dz = \int_0^\pi z_2(t)\,z_2'(t)\,dt$$

$$\Rightarrow \int_{\gamma_2} f(z)dz = \int_0^\pi \left(1/2 + 1/2\, e^{i(\pi-t)}\right)\left(-1/2\, i\, e^{i(\pi-t)}\right)dt$$

$$= -1/4\, i \int_0^\pi \left(e^{i(\pi-t)} - e^{2i(\pi-t)}\right)dt$$

$$= -\frac{1}{4} i \left(\frac{-1}{i} e^{i(\pi-t)} + \frac{e^{2i(\pi-t)}}{2i} \right)\Bigg|_0^\pi$$

$$\Rightarrow \int_{\gamma_2} f(z)\,dz = \frac{-i}{4}\left(\frac{-1}{i}(1-e^{i\pi}) + \frac{1}{2i}(1-e^{2i\pi})\right)$$

$$= \frac{-i}{4}\left(\frac{-1}{i}\cdot 2\right)$$

$$= 1/2.$$

Example:

Compute $\int_{\gamma_r}(z-z_0)^n\,dz$ for $n \in \mathbb{Z}$ and γ_r any circle of radius r centered at z_0.

We can parametrize γ_r by

$$z_r(t) = z_0 + re^{it}, \quad 0 \le t \le 2\pi$$

$$\Rightarrow z_r'(t) = rie^{it}$$

$$\Rightarrow \int_{\gamma_r}(z-z_0)^{-n}\,dz = \int_0^{2\pi}(re^{it})^{-n}\,rie^{it}\,dt = \int_0^{2\pi}\frac{ri}{r^n e^{nit}}e^{it}\,dt$$

$$\Rightarrow \int_{\gamma_r}(z-z_0)^{-n}\,dz = \int_0^{2\pi} i r^{1-n} e^{it(1-n)}\,dt$$

$$= i\begin{cases} r^{1-n}\frac{1}{i(1-n)}e^{it(1-n)}\Big|_0^{2\pi}, & n \ne 1 \\ 1\,\Big|_0^{2\pi}, & n = 1 \end{cases}$$

$$= i\begin{cases} r^{1-n}0, & n \ne 1 \\ 2\pi, & n = 0 \end{cases}$$

$$\Rightarrow \int_{\gamma_r}(z-z_0)^{-n}\,dz = \begin{cases} 2\pi i, & n = 1 \\ 0, & \text{o.w.} \end{cases}$$

Theorem - Suppose f is continuous on a domain D with antiderivative F. Then for any contour in D connecting $\alpha \in \mathbb{C}$ to $\beta \in \mathbb{C}$:

$$\int_\gamma f(z)\,dz = F(\beta) - F(\alpha).$$

proof:

Let $z_1 : [a,b] \to \mathbb{C}$ be a parametrization of γ. Therefore,

$$\int_\gamma f(z)\,dz = \int_a^b f(z_1(t))\, z_1'(t)$$

However, by the chain rule

$$\frac{d}{dt} F(z_1(t)) = F'(z_1(t)) z_1'(t) = f(z_1(t)) z_1'(t).$$

Therefore,

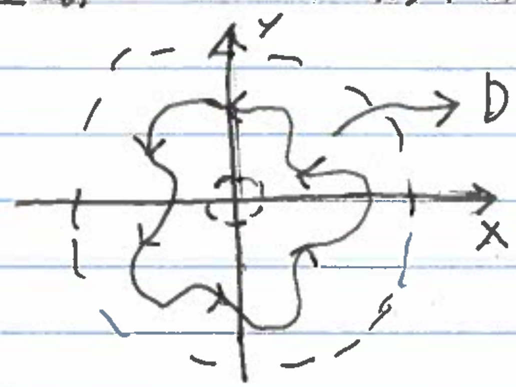
$$\int_\gamma f(z)\,dz = \int_a^b \frac{d}{dt} F(z_1(t))\,dt = F(\beta) - F(\alpha).$$

Corollary - If f is continuous in D with continuous antiderivative F then for all closed curves:

$$\int_\gamma f(z)\,dz = 0.$$

Example:

$$\int_\gamma \frac{1}{(z-z_0)^n}\,dz = 0 \quad \text{if } n \neq 1 \text{ and } n \in \mathbb{Z}:$$



$f(z) = \frac{1}{(z-z_0)^n}$ is analytic in D with antiderivative

$F(z) = \frac{(z-z_0)^{-n+1}}{1-n}$

Note, if $f(z) = 1/z-z_0$ then $F(z) = \mathrm{Log}(z-z_0)$ which is not analytic.