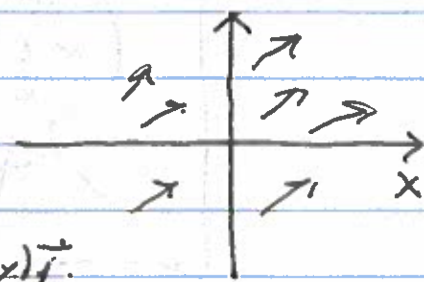
Lecture 18: Cauchy Integral Theorem

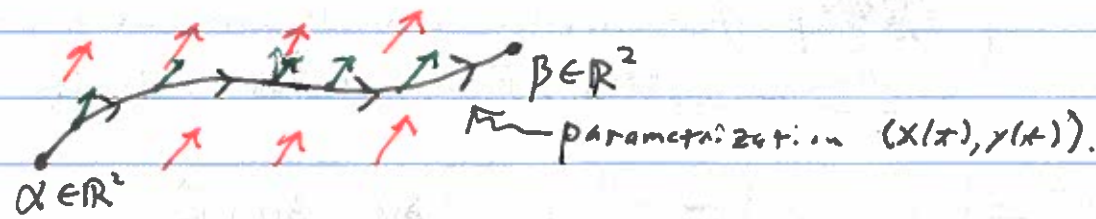
Vector Calculus Review

$F: \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field.



Assigns to a coordinate (x,y) a vector with components $u(x,y)$, $v(x,y)$:

$$F(x,y) = \langle u(x,y), v(x,y) \rangle = u(x,y)\vec{i} + v(x,y)\vec{j}.$$



$$\Rightarrow \int_\gamma \underbrace{F(x,y)\cdot d\vec{r}}_{\substack{\text{projection of}\\ \text{vector field onto}\\ \text{tangent vector}}} = \int_a^b F(x(t),y(t)) \cdot \langle \tfrac{dx}{dt}, \tfrac{dy}{dt} \rangle \, dt$$

$$\Rightarrow \int_\gamma F(x,y)\cdot d\vec{r} = \int_a^b u(x(t),y(t))\frac{dx}{dt}\,dt + \int_a^b v(x(t),y(t))\frac{dy}{dt}\,dt.$$

* Measures total influence of vector field along curve.

If there exists $V: \mathbb{R}^2 \to \mathbb{R}$ such that $F = \nabla V$

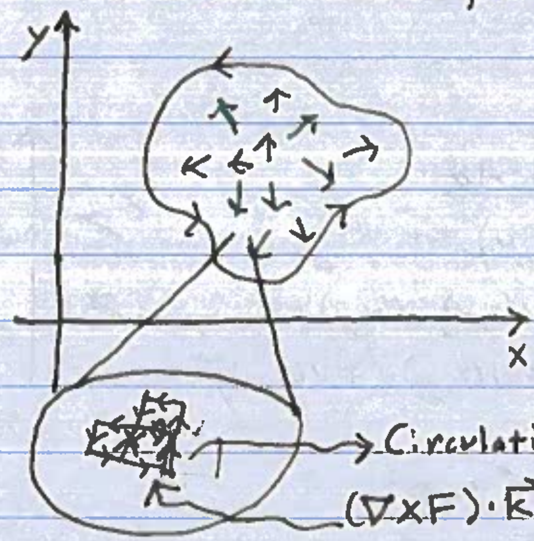
$$\Rightarrow F = \langle \tfrac{\partial V}{\partial x}, \tfrac{\partial V}{\partial y} \rangle$$

then

$$\int_\gamma F(x,y)\,dr = \int_a^b \left(\frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}\right)dt$$
$$= \int_a^b \frac{d}{dt} V(x(t),y(t))\,dt$$
$$= V(x(b),y(b)) - V(x(a),y(a))$$
$$= V(\beta) - V(\alpha) \quad \text{(potential difference)}$$

(path independence)

Lets look at a closed loop:



$$\iint_D \nabla \times F \cdot \vec{k}\, dA = \int_\gamma F \cdot d\vec{r}$$

Circulation inside of D. — Circulation on boundary

Circulation cancels out except at the boundary

$(\nabla \times F) \cdot \vec{k} =$ circulation density

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ U & V & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)\vec{k}$$

$$\Rightarrow \int_\gamma F \cdot dr = \underbrace{\int_a^b \left(U(x,y)\frac{dx}{dt}\,dt + V(x,y)\frac{dy}{dt}\,dt \right) = \iint_D \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) dA}_{\text{Classic math Green's theorem.}} = \iint_D \nabla \times \vec{F} \cdot \vec{k}\, dA$$

Big Theorem -

- $\nabla \times \nabla V = 0$ and $\nabla \times F = 0$ if and only if there exists V such that $F = \nabla V$.
- If $f: \mathbb{C} \to \mathbb{C}$ is analytic on a simply connected domain D and Γ is any closed curve forming the boundary of Γ then

$$\int_\Gamma f(z)\,dz = 0$$

~~proof~~:

Let $f(z) = u(x,y) + iv(x,y)$. Then,

$$\int_\Gamma f(z)\,dz = \int_\Gamma (u(x,y) + iv(x,y))\left(\frac{dx}{dt} + i\frac{dy}{dt}\right) dt$$

$$= \int_\Gamma \left(u(x,y)\frac{dx}{dt} - v(x,y)\frac{dy}{dt}\right) dt + i\int_\Gamma \left(v(x,y)\frac{dx}{dt} + u(x,y)\frac{dy}{dt}\right) dt$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA$$

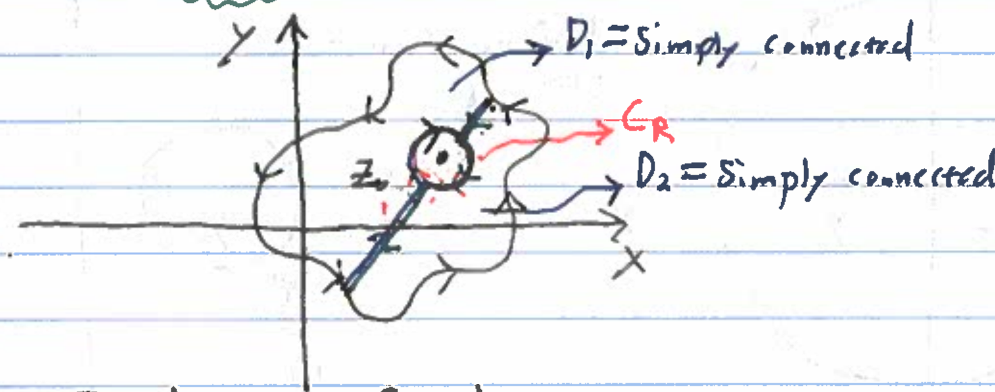
$= 0$, by Cauchy-Riemann equations!

Examples:

1. Let γ be any closed contour enclosing $z=z_0$. What is

$$\int_\gamma \frac{1}{z-z_0}\,dz\ ?$$

Not analytic at z_0!!

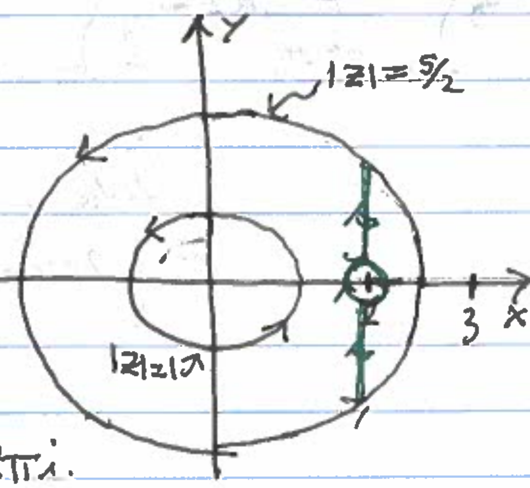


$$0=\int_\gamma \frac{1}{z-z_0}\,dz-\int_{C_R}\frac{1}{z-z_0}\,dz + \text{integrals which cancel}$$

$$\Rightarrow \int_\gamma \frac{1}{z-z_0}\,dz=\int_{C_R}\frac{1}{z-z_0}\,dz=2\pi i$$

* The original contour was deformed to a circle about z_0.

2. $$\int_{|z|=1}\frac{1}{(z-3)(z-2)}\,dz=0$$



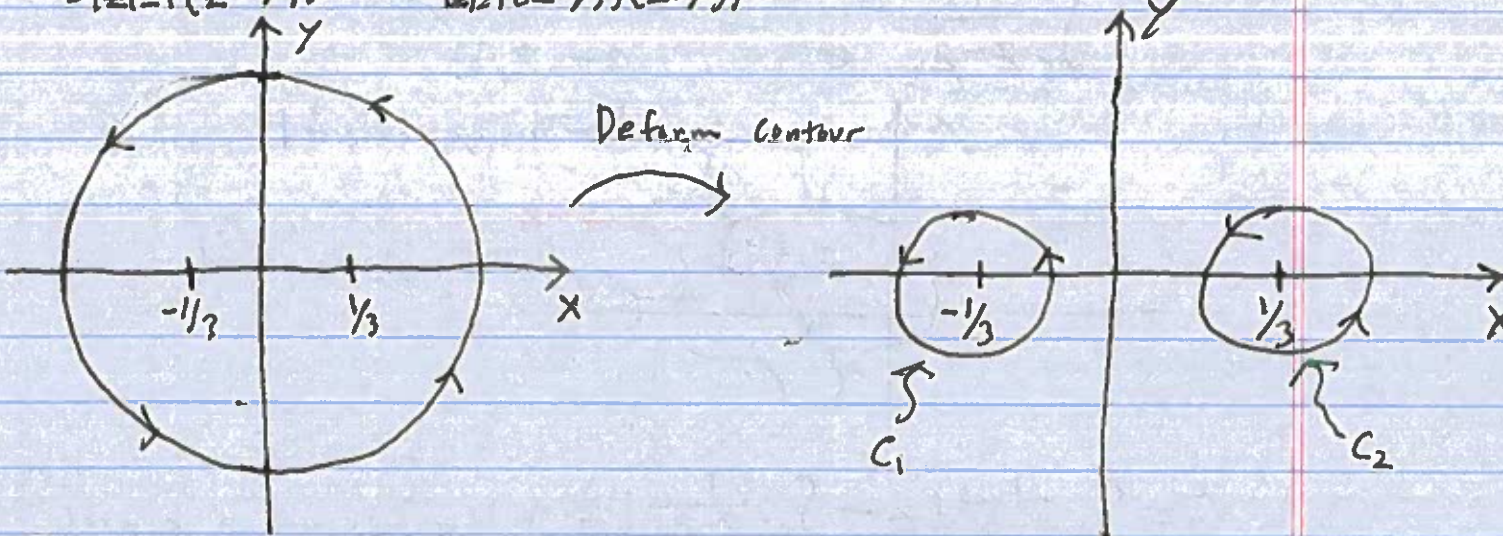
3. $$\int_{|z|=5/2}\frac{1}{(z-3)(z-2)}\,dz=\int_{|z|=5/2}\left(\frac{1}{z-3}-\frac{1}{z-2}\right)dz$$

$$=\int_{|z|=5/2}-\frac{1}{z-2}\,dz=-2\pi i.$$

4. $$\int_{|z|=4}\frac{1}{(z-3)(z-2)}\,dz=\int_{|z|=4}\frac{1}{z-3}\,dz-\int_{|z|=4}\frac{1}{z-2}\,dz=2\pi i-2\pi i=0.$$

Example:

$$\int_{|z|=1} \frac{z}{(z^2-1/9)}\,dz = \int_{|z|=1} \frac{z}{(z-1/3)(z+1/3)}\,dz$$



$$\int_{|z|=1} \frac{z}{(z-1/3)(z+1/3)}\,dz = \int_{|z|=1} \left(\frac{A}{z-1/3} + \frac{B}{z+1/3}\right)dz = \int_{|z|=1} \frac{A(z+1/3)+B(z-1/3)}{(z-1/3)(z+1/3)}\,dz$$

$$\Rightarrow A+B=1 \Rightarrow A=1/2,\ B=1/2$$
$$A-B=0$$

$$\Rightarrow \int_{|z|=1} \frac{z}{z^2-1/9}\,dz = \frac{1}{2}\int_{|z|=1} \frac{1}{z-1/3}\,dz + \frac{1}{2}\int_{|z|=1} \frac{1}{z+1/3}\,dz$$

$$= \frac{1}{2}\int_{\gamma_1} \frac{1}{z-1/3}\,dz + \frac{1}{2}\int_{\gamma_1} \frac{1}{z+1/3}\,dz + \frac{1}{2}\int_{\gamma_2} \frac{1}{z+1/3}\,dz + \frac{1}{2}\int_{\gamma_2} \frac{1}{z+1/3}\,dz$$

$$= \frac{1}{2}\cdot 2\pi i + 0 + \frac{1}{2}\,2\pi i + 0 = 2\pi i.$$