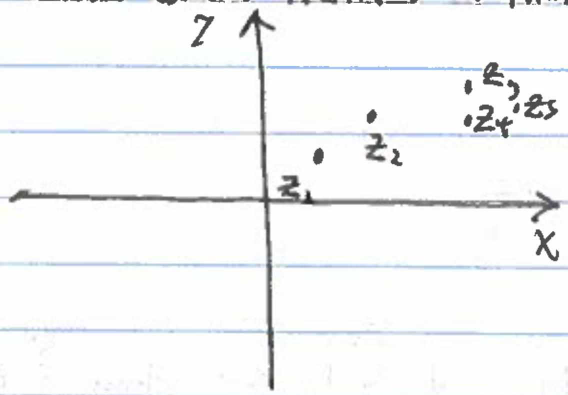
Lecture 20: Sequences and Series

Definition - A sequence $z_n \in \mathbb{C}$ is Cauchy if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m,n \geq N \Rightarrow |z_n - z_m| < \varepsilon$.



$\rightarrow$ Cauchy sequence is about the tail of a sequence clustering.

Theorem (Completeness of $\mathbb{C}$) - A sequence $z_n \in \mathbb{C}$ is convergent if and only if it is Cauchy.

Definition - A series is a sum of the form

$$S_n = \sum_{j=0}^{n} c_j.$$

We say the series converges if S_n converges and write

$$S = \lim_{n\to\infty} S_n = \sum_{j=0}^{\infty} c_j.$$

Theorem - If $\sum_{n\geq 0}^{\infty} |c_j|$ converges then $\sum_{j\geq 0}^{\infty} c_j$ converges.

~~proof~~ -

Let $S_n = \sum_{j=0}^{n} c_j$ and $T_n = \sum_{j=0}^{n} |c_j|$. Therefore, for all $m,n \in \mathbb{N}$

$$|S_n - S_m| = \left|\sum_{j=0}^{n} c_j - \sum_{j=0}^{m} c_j\right|$$

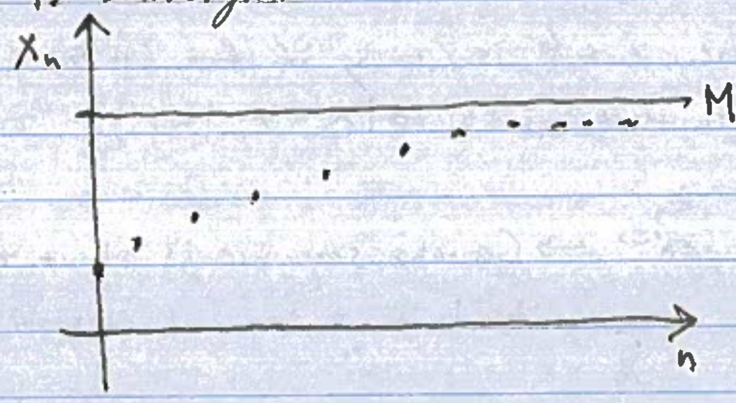
$$= \left|\sum_{j=n+1}^{m} c_j\right|$$

$$\leq \sum_{j=n}^{m} |c_j|$$

$$= |T_m - T_n|.$$

Since T_n is Cauchy it follows that S_n is Cauchy as well.

<u>Theorem</u> - If $x_n \in \mathbb{R}$ is monotone increasing and bounded above then x_n is convergent.



<u>Theorem</u> - If c_j satisfy $|c_j| \leq M_j$ and $\sum_{j=0}^{\infty} M_j < \infty$ then $\sum_{j=0}^{\infty} c_j$ converges.

proof:

Let $S_n = \sum_{j=0}^{n} |c_j|$. Therefore, S_n is monotone increasing and satisfies

$$S_n \leq \sum_{j=0}^{\infty} M_j.$$

Consequently, S_n converges and thus $\sum_{j=0}^{\infty} c_j$ converges.

<u>Theorem</u> - The series $\sum_{j=0}^{\infty} c^j$ converges to $1/1-c$ if $|c|<1$.

proof:

Let $S_n = \sum_{j=0}^{n} c^j$. First we note that:

$$S_n = 1 + c + c^2 + \dots + c^n$$
$$\Rightarrow cS_n = c + c^2 + c^3 + \dots + c^{n+1}$$
$$\Rightarrow cS_n = S_n + c^{n+1} - 1$$
$$\Rightarrow S_n = \frac{c^{n+1}-1}{c-1},$$

Now,

$$\left| S_n - \frac{1}{c-1} \right| = \left| \frac{-c^{n+1}}{1-c} \right| = \frac{|c|^{n+1}}{|1-c|}$$

By the squeeze theorem

$$\lim_{n \to \infty} \left| S_n - \frac{1}{1-c} \right| = 0.$$

Example:

Determine the region in which

$$\sum_{n=0}^{\infty} \frac{1}{(4+2z)^n}$$

is convergent and find the sum. Let $w = (4+2z)^{-1}$. This sum converges when

$$|(4+2z)^{-1}| < 1 \Rightarrow |(4+2z)| > 1 \Rightarrow |2+z| > 1/2.$$

The sum is then given by:

$$\frac{1}{1-\frac{1}{4+2z}} = \frac{4+2z}{3+2z}$$

Example:

Show that $\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n}$ converges for all θ and find the sum.

Note,

$$\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n} = Re\left(\sum_{n=0}^{\infty} \frac{e^{in\theta}}{2^n}\right) = Re\left(\sum_{n=0}^{\infty}\left(\frac{e^{i\theta}}{2}\right)^n\right) = Re\left(\frac{1}{1-e^{i\theta}/2}\right)$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n} = Re\left(\frac{1+e^{-i\theta}/2}{3/2}\right) = \frac{2}{3}(1+\cos(\theta)).$$

Theorem – Suppose that the terms of the series $\sum_{j=1}^{\infty} c_j$ have the property that $|c_{j+1}/c_j| \to L$ as $j \to \infty$. Then, the series converges if $L < 1$ and diverges if $L > 1$.

Example:

Show that the series $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$ converges for all z.

Solution:

Let $\rho_n = \left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n} \right| = \frac{|z|}{n+1}$. Therefore, $\lim_{n \to \infty} \rho_n = 0$

and thus the series converges for all z.

Theorem – $\sum_{n=0}^{\infty} (z/z_0)^j = \frac{1}{1 - z/z_0} = \frac{z_0}{z_0 - z}$ if $|z/z_0| < 1$.