Lecture 23: The residue Theorem

Definition - A point z_0 is called a zero of order m for a function f if

- f is analytic at z_0
- $f(z_0)=0$
- $f^{(j)}(z_0)=0$ if $j<m$
- $f^{(m)}(z_0)\neq 0$

Definition - Let f have an isolated singularity at z_0 with Laurent series

$$\sum_{j=-\infty}^{\infty} a_j (z-z_0)^j$$

We say f has a pole of order m if

- $a_j=0$ for $j<-m$
- $a_{-m}\neq 0$

Theorem - A function has a pole of order m at z_0 if and only if

$$f(z)=\frac{g(z)}{(z-z_0)^m}$$

where g is analytic at z_0 and $g(z_0)\neq 0$.

Example:

$$\cot^2(z)=\frac{\cos^2(z)}{\sin^2(z)}$$

$$=\frac{\cos^2(z)}{(z-z^3/3!+\dots)(z-z^3/3!+\dots)}$$

$$=\frac{\cos^2(z)}{z^2(1-z^2/3!+\dots)(1-z^2/3!+\dots)}$$

$$\Rightarrow \cot^2(z) = \frac{\cos^2(z)}{z^2(1-g(z))}$$

(geometric series is like factoring the denominator)

$$= \frac{\cos^2(z)\,(1+g(z)+g(z)^2+\dots)}{z^2}$$

Numerator $\neq 0$ at $z=0 \Rightarrow$ pole of order 2.

Essential Singularity - $f(z)$ has an essential singularity at z_0 if the Laurent series

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-z_0)^j$$

satisfies $a_j \neq 0$ for an infinite number of $j<0$.

Definition - If z_0 is an isolated singularity of f then the coefficient a_{-1} of the Laurent series centered at z_0 is called the residue of f at z_0 and is denoted by

$$\mathrm{Res}(f, z_0).$$

Example: Find the residues of

$$f(z) = z e^{1/z} + \frac{3z}{1+z^2}$$

and compute

$$\int_{|z|=1} f(z)\,dz, \int_{|z|=1/2} f(z)\,dz, \int_{|z-i|=3/2} f(z)\,dz$$