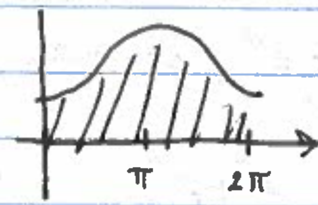
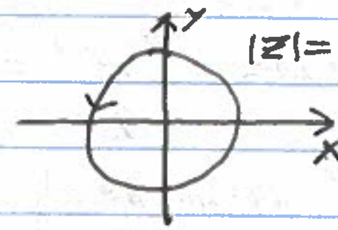
Lecture 24: Trigonometric Integrals

Example:

$$\int_0^{2\pi} \frac{d\theta}{10+8\cos\theta}$$





On $z=1$ we have

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$$

$$dz = ie^{i\theta}d\theta = iz\,d\theta$$

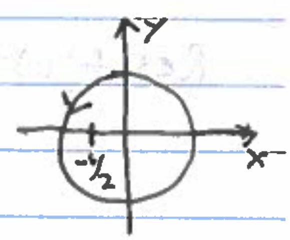
$$d\theta = \frac{1}{iz}dz$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{10+8\cos\theta} = \oint_{|z|=1} \frac{1}{iz(10+4(z+z^{-1}))}dz$$

$$= \oint_{|z|=1} \frac{1}{i(10z+4z^2+4)}dz$$

$$= \oint_{|z|=1} \frac{1}{i2(2z^2+5z+2)}dz$$

$$= \oint_{|z|=1} \frac{1}{i2(2z+1)(z+2)}dz$$



$$= 2\pi i \,\mathrm{Res}(f; -\tfrac{1}{2})$$

$$= \frac{2\pi i}{i2}\cdot\frac{1}{-\frac{1}{2}+2}$$

$$= \frac{\pi}{3}.$$

Example:

$$\int_0^{2\pi} \frac{\cos(2\theta)}{2+\cos\theta}\, d\theta, \quad z = e^{i\theta}$$

$$dz = i e^{i\theta} d\theta = i z\, d\theta$$

$$\Rightarrow \cos 2\theta = \frac{e^{i2\theta}+e^{-i2\theta}}{2} = \frac{z^2+z^{-2}}{2}, \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$$

$$\Rightarrow \int_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta}\, d\theta = \oint_{|z|=1} \frac{1}{2i} \frac{z^2+z^{-2}}{z(2+\frac{z+z^{-1}}{2})}\, dz$$

$$= \oint_{|z|=1} \frac{z^2+z^{-2}}{i(4z++z^2+1)}\, dz$$

$$= \int_{|z|=1} \frac{1+z^4}{iz^2(z^2+4z+1)}\, dz$$

Solve $z^2+4z+1=0$

$$\Rightarrow z = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$

$$\Rightarrow \int_0^{2\pi} \frac{\cos(2\theta)}{2+\cos\theta}\, d\theta = \int_{|z|=1} \frac{1+z^4}{iz^2(z+2-\sqrt{3})(z+2+\sqrt{3})}\, dz$$

$$\mathrm{Res}(f;0) = \lim_{z\to 0} \frac{d}{dz} \frac{1+z^4}{i(z^2+4z+1)} = \lim_{z\to 0} \frac{(z^2+4z+1)4z^3-(1+z^4)(2z+4)}{i(z^2+4z+1)^2}$$

$$= \frac{-4}{i}$$

$$\mathrm{Res}(f; -2+\sqrt{3}) = \frac{1+(-2+\sqrt{3})^4}{i(2\sqrt{3})} = \frac{7}{\sqrt{3}i}$$

$$\Rightarrow \int_0^{2\pi} \frac{\cos(2\theta)}{2+\cos(\theta)}\, d\theta = -8\pi + \frac{14\pi}{\sqrt{3}}.$$