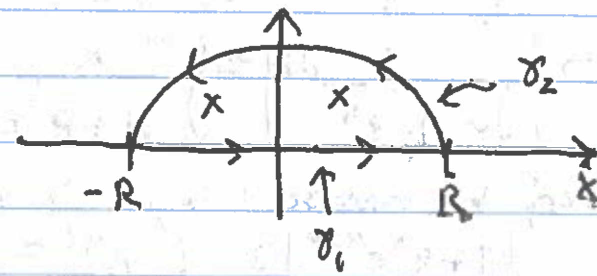
Lecture 25: Improper Integrals over $\mathbb{R}$

Example:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx$$



Let $\gamma = \gamma_1 \cup \gamma_2$. Then,

$$\int_\gamma \frac{1}{1+z^4}\,dz = \int_{-R}^{R} \frac{1}{1+x^4}\,dx + \int_{\gamma_2} \frac{1}{1+z^4}\,dz$$

$$1+z^4 = 0 \Rightarrow z^4 = -1$$

$$\Rightarrow z^4 = e^{i\pi + 2n\pi}$$

$$\Rightarrow z = e^{i\pi/4 + n\pi/2}$$

Simple poles at $z_1 = e^{i\pi/4}$, $z_2 = e^{i3\pi/4}$

$$\frac{1}{z^4+1} = \frac{1}{(z-e^{-i\pi/4})(z-e^{-i3i\pi/4})(z-e^{i\pi/4})(z-e^{3i\pi/4})}$$

$$\mathrm{Res}(f; e^{i\pi/4}) = \frac{1}{(e^{i\pi/4}-e^{-i\pi/4})(e^{i\pi/4}-e^{-i3i\pi/4})(e^{i\pi/4}-e^{3i\pi/4})}$$

$$= \frac{1}{e^{3i\pi/4}(1-e^{-i\pi/2})(1-e^{-i\pi})(1-e^{2i\pi/4})}$$

$$= \frac{1}{4e^{3i\pi/4}} = \frac{1}{4}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$\mathrm{Res}(f; e^{3i\pi/4}) = \frac{1}{4}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow \int_\gamma \frac{1}{1+z^4}\,dz = \frac{2\pi i}{4}\left(-i\sqrt{2}\right)$$

$$= \frac{\pi}{\sqrt{2}}$$

Now, $\int_{\gamma_2} \frac{1}{1+z^4}\,dz = \int_0^{2\pi} \frac{Rie^{i\theta}}{1+R^4e^{4i\theta}}\,d\theta$

$$\Rightarrow \left| \int_{\gamma_2} \frac{1}{1+z^4}\,dz \right| \leq \int_0^{2\pi} \frac{R}{|1+R^4e^{4i\theta}|}\,d\theta$$

However, by reverse triangle inequality

$$|1+R^4e^{4i\theta}| \geq |1-R^4| = R^4-1 \quad \text{since } R>1.$$

$$\Rightarrow \left| \int_{\gamma_2} \frac{1}{1+z^4}\,dz \right| \leq \int_0^{2\pi} \frac{R}{R^4-1}\,d\theta = \frac{2\pi R}{R^4-1}.$$

Therefore, by the squeeze theorem

$$\lim_{R\to\infty} \int_{\gamma_2} \frac{1}{1+z^4}\,dz = 0.$$

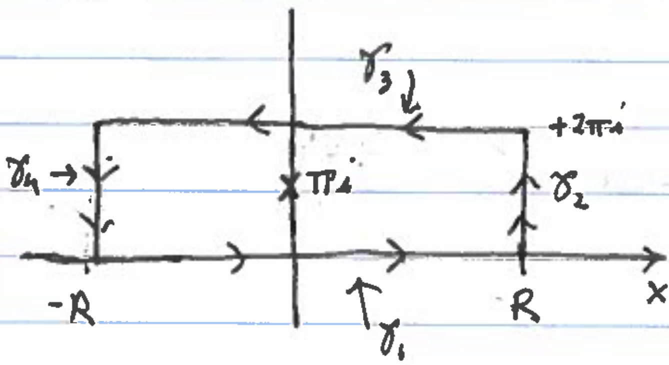
Therefore,

$$\frac{\pi}{\sqrt{2}} = \lim_{R\to\infty} \int_{\gamma} \frac{1}{1+z^4}\,dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4}\,dx + \lim_{R\to\infty} \int_{\gamma_2} \frac{1}{1+z^4}\,dz$$

$$\Rightarrow \frac{\pi}{\sqrt{2}} = \int_{\infty}^{\infty} \frac{1}{1+x^4}\,dx.$$

Example:

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx, \quad 0<a<1$$



γ_1

$$\int_{\gamma_1} \frac{e^{az}}{1+e^z}\,dz = \int_{-R}^{R} \frac{e^{ax}}{1+e^x}\,dx$$

γ_2:

$z = R+iy, \quad y \in [0, 2\pi], \quad dz = i\,dy$

$$\int_{\gamma_2} \frac{e^{az}}{1+e^z}\,dz = \int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{R+iy}}\,i\,dy$$

$$\Rightarrow \left|\int_{\gamma_2} \frac{e^{az}}{1+e^z}\,dz\right| \le \int_0^{2\pi} \frac{e^{aR}}{|1+e^R e^{iy}|}\,dy$$

$$\le \int_0^{2\pi} \frac{e^{aR}}{e^R-1}\,dy$$

$$= \frac{2\pi e^{aR}}{e^R-1}$$

$$\Rightarrow \lim_{R\to\infty} \int_{\gamma_2} \frac{e^{az}}{1+e^z}\,dz = 0.$$

γ_3: $z = x + 2\pi i$, $dz = dx$, $x \in [R, -R]$

$$\int_{\gamma_3} \frac{e^{az}}{1+e^z}\,dz = \int_R^{-R} \frac{e^{2\pi i a}e^{ax}}{1+e^{2\pi i}e^x}\,dx = -e^{2\pi i a}\int_{-R}^{R} \frac{e^{ax}}{1+e^x}\,dx = -e^{2\pi i a}\int_{\gamma_1} \frac{e^{az}}{1+e^z}\,dz$$

γ_4:

$$\lim_{R\to\infty} \int_{\gamma_4} \frac{e^{az}}{1+e^z}\,dz = 0 \text{ as well}$$

The residue at $z = \pi i$ is given by:

$$\frac{e^{az}}{1+e^z} = \frac{e^{az}}{1+(-1+e^z|_{\pi i}(z-\pi i)+\dots)} = \frac{e^{az}}{-1\cdot(z-\pi i)+a(z-\pi i)^2+\dots}$$

$$\Rightarrow \frac{e^{az}}{1+e^z} = \frac{e^{az}}{-(z-\pi i)(1-f(z-\pi i))} = \frac{e^{az}}{-(z-\pi i)}(1+f(z-\pi i)+f^2(z-\pi i)+\dots)$$

Therefore,

$$\operatorname{Res}\left(\frac{e^{az}}{1+e^z}\right) = -e^{a\pi i}$$

Putting all of the pieces together

$$\lim_{R\to\infty} \int_{\Gamma} \frac{e^{az}}{1+e^z}\,dz = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx - e^{2\pi a i}\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = -2\pi i e^{a\pi i}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{2\pi i e^{a\pi i}}{e^{2\pi a i}-1} = \frac{2\pi i}{e^{\pi a i}-e^{-\pi a i}} = \frac{\pi}{\sin(\pi a)}.$$