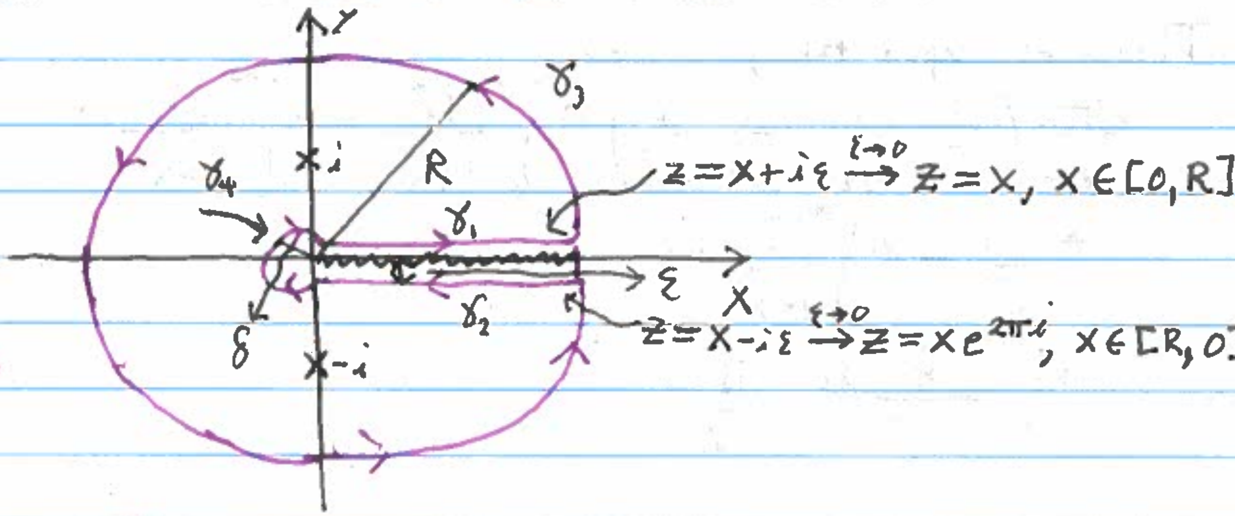
Lecture 23: Multivalued Functions

Example:

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$$

The idea is to take the $[0, 2\pi)$ branch of $\sqrt{x}$ since we cannot allow contours to cross branch cuts.



$$f(z) = \frac{\sqrt{z}}{1+z^2}$$

1. $$\int_{\gamma_1} f(z)\,dz = \int_0^R \frac{\sqrt{x}}{1+x^2}\,dx$$

2. $$\int_{\gamma_2} f(z)\,dz = \int_R^0 \frac{\sqrt{x}\,e^{\pi i}}{1+x^2}\,dx = -\int_R^0 \frac{\sqrt{x}}{1+x^2}\,dx = \int_0^R \frac{\sqrt{x}}{1+x^2}\,dx$$

3. $$\left|\int_{\gamma_3} f(z)\,dz\right| \le \int_0^{2\pi} \frac{R^{3/2}}{R^2-1}\,d\theta = \frac{2\pi R^{3/2}}{R^2-1}.$$

Therefore, $\lim_{R\to\infty} \int_{\gamma_3} f(z)\,dz = 0.$

4. $$\int_{\gamma_4} f(z)\,dz = \int_{2\pi}^0 \frac{\delta^{1/2} e^{i\theta/2}\,\delta i e^{i\theta}}{1+\delta^2 e^{2i\theta}}\,d\theta$$

$$\Rightarrow \lim_{\delta\to 0} \int_{\gamma_4} f(z)\,dz = 0.$$

Furthermore,

$$\mathrm{Res}(f;i) = \lim_{z\to i} \frac{\sqrt{z}}{i+z} = \left(\tfrac{1}{\sqrt{2}} + \tfrac{i}{\sqrt{2}}\right)/2i$$

$$\mathrm{Res}(f;-i) = \lim_{z\to -i} \frac{\sqrt{z}}{i-z} = \left(\tfrac{1}{\sqrt{2}} - \tfrac{i}{\sqrt{2}}\right)/2i$$

Therefore, upon taking limits

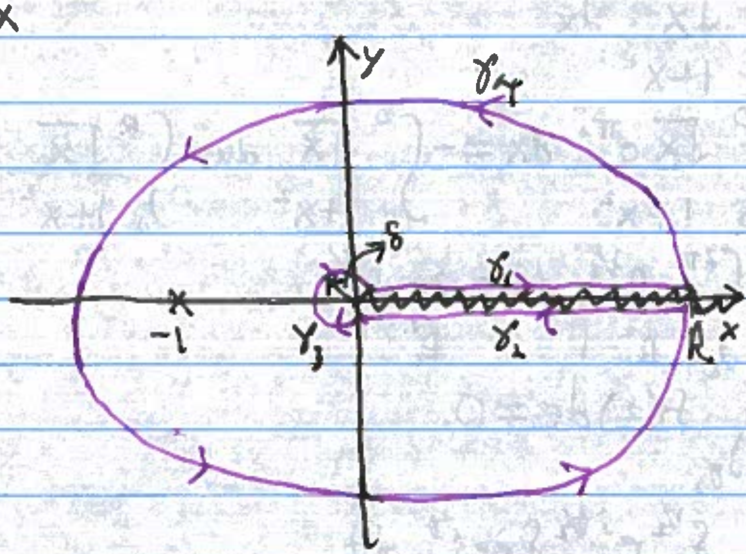
$$\frac{2\pi i}{2i}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx + \int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$$

$$\Rightarrow \frac{2\pi}{\sqrt{2}} = 2\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$$

$$\Rightarrow \int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx = \frac{\pi}{\sqrt{2}}.$$

Example:

$$\int_0^\infty \frac{x^\lambda}{1+x}\,dx, \quad -1<\lambda<0$$



$$\mathrm{Res}(f;-1) = z^\lambda\big|_{-1} = (-1)^\lambda = e^{i\pi\lambda}$$

1. $\displaystyle\int_{\gamma_1} f(z)\,dz = \int_0^R \frac{x^\lambda}{1+x}\,dx$

2. $\displaystyle\int_{\gamma_2} f(z)\,dz = \int_R^0 \frac{e^{2\pi i\lambda} x^\lambda}{1+x}\,dx = -e^{2\pi i\lambda}\int_0^R \frac{x^\lambda}{1+x}\,dx$

3. $$\left|\int_{\gamma_3} f(z)\,dz\right| \leq \int_0^{2\pi} \left|\frac{\delta^\lambda e^{\lambda i\theta}\,\delta i e^{i\theta}}{1+\delta e^{i\theta}}\right| d\theta$$
$$= \int_0^{2\pi} \frac{\delta^{1+\lambda}}{1-\delta}\,d\theta$$
$$= \frac{2\pi\,\delta^{1+\lambda}}{1-\delta}$$

and thus $\displaystyle\lim_{\delta\to 0}\int_{\gamma_3} f(z)\,dz = 0.$

4. $$\left|\int_{\gamma_4} f(z)\,dz\right| \leq \int_0^{2\pi} \frac{R^{1+\lambda}}{R-1}\,d\theta = \frac{2\pi R^{1+\lambda}}{R-1}.$$

and thus since $1+\lambda < 1$ it follows that

$$\lim_{R\to\infty}\int_{\gamma_4} f(z)\,dz = 0.$$

Putting everything together we have

$$e^{i\pi\lambda} = \lim_{R\to\infty}\lim_{\delta\to 0}\int_\Gamma f(z)\,dz$$
$$= \int_0^\infty \frac{x^\lambda}{1+x}\,dx - e^{2\pi i\lambda}\int_0^\infty \frac{x^\lambda}{1+x}\,dx$$

$$\Rightarrow \int_0^\infty \frac{x^\lambda}{1+x}\,dx = \frac{e^{i\pi\lambda}\cdot 2\pi i}{1-e^{2\pi i\lambda}} = \frac{2\pi i}{e^{-i\pi\lambda}-e^{i\pi\lambda}} = \frac{\pi}{\sin(-\lambda\pi)}$$