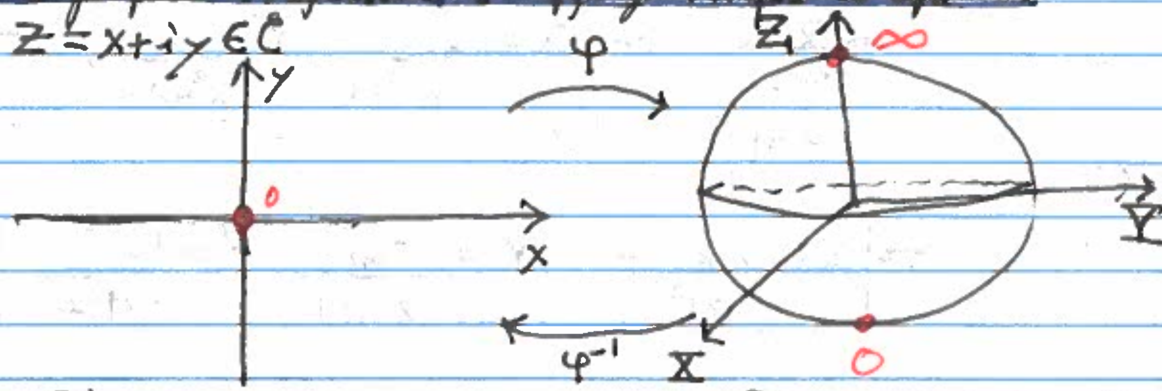
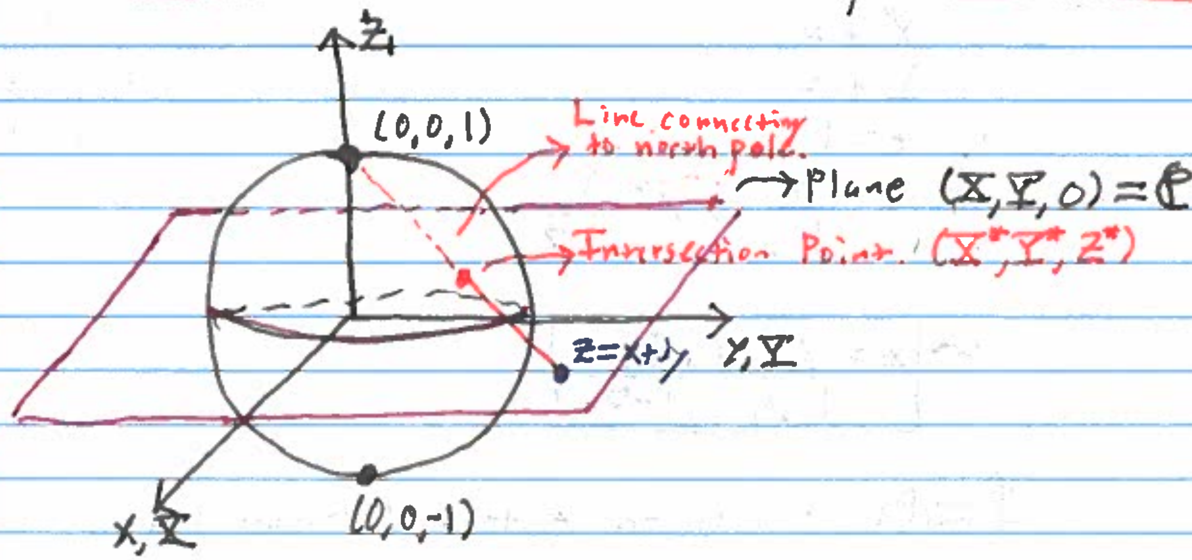
# Lecture 7: The Riemann Sphere

Where is $\infty$ in $\mathbb{C}$??

Stereographic Projection (Mapping Planes to Spheres)




Plane | Sphere of Radius 1




Parametrization of line connecting $(x,y,0)$ to $(0,0,1)$:

$\vec{X} = (x,y,0), \quad \vec{N} = (0,0,1)$

$\alpha(t): [0,1] \to \mathbb{R}^3$ defined by

$\alpha(t) = (1-t)\vec{N} + t\vec{X}$

$\Rightarrow X(t) = tx$

$Y(t) = ty$

$Z(t) = 1-t$

Intersection occurs when $X^2+Y^2+Z^2=1$

$\Rightarrow t^2x^2+t^2y^2+(1-t)^2=1$:

$\Rightarrow t^2x^2+t^2y^2+1-2t+t^2=1$

$\Rightarrow t^2x^2+t^2y^2+t^2-2t=0$

$\Rightarrow t(tx^2+ty^2+t-2)=0$

Therefore,

$$t = 0 \quad \text{or} \quad t = \frac{2}{x^2+y^2+1}.$$

$$\Rightarrow \begin{array}{l} X=0 \\ Y=0 \\ Z=1 \end{array} \quad \text{or} \quad X = \frac{2x}{x^2+y^2+1},\ Y = \frac{2y}{x^2+y^2+1},\ Z_1 = 1 - \frac{2}{x^2+y^2+1}$$

$$\Rightarrow \boxed{X = \frac{2\operatorname{Re}(z)}{|z|^2+1},\ Y = \frac{2\operatorname{Im}(z)}{|z|^2+1},\ Z_1 = \frac{|z|^2-1}{|z|^2+1}}$$

We can also go backwards:

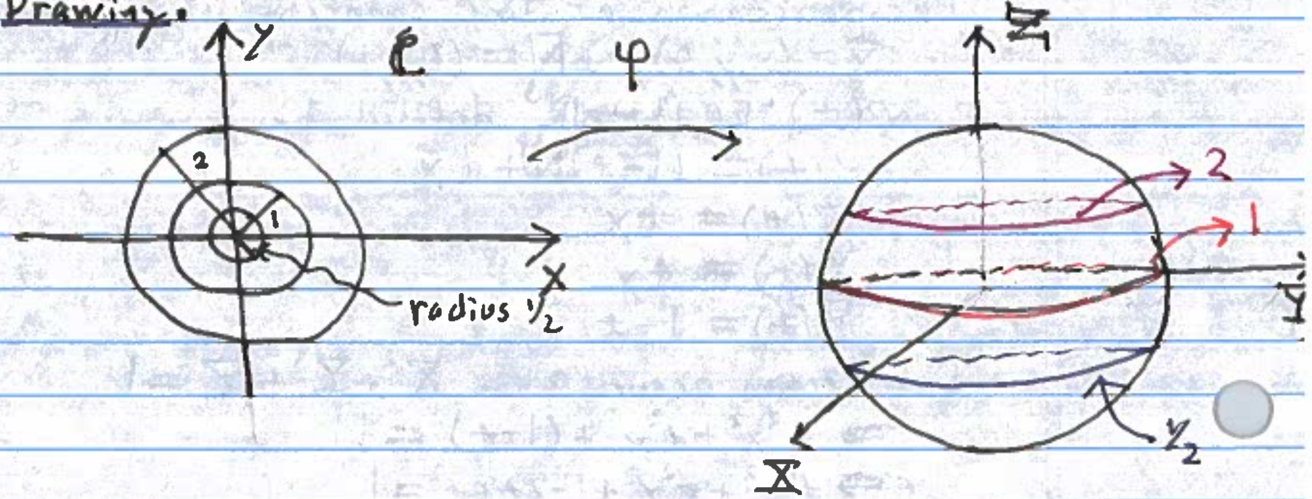
$$X/t = x$$
$$Y/t = y$$
$$t = 1 - Z_1$$
$$\Rightarrow x = \frac{X}{1-Z_1},\ y = \frac{Y}{1-Z_1}$$

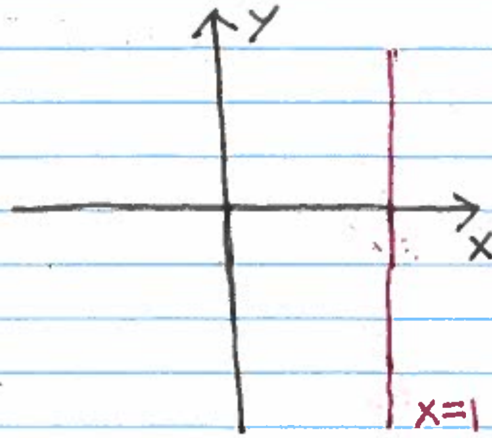
$$\Rightarrow \boxed{z = \frac{X}{1-Z_1} + i\frac{Y}{1-Z_1}}$$

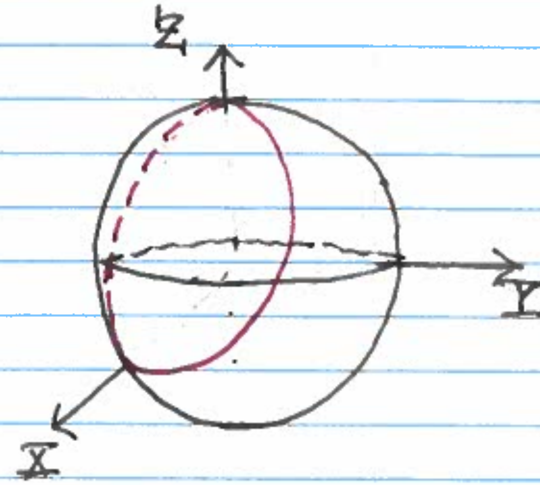
Bijection!!

$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended complex plane

<u>Drawing:</u>






## Distance:

Look at text for derivation:

$$z = x+iy \quad , \quad w = u+iv$$

$$\downarrow \varphi \qquad\qquad \downarrow \varphi$$

$$\overline{X} = (X, Y, Z) \qquad (U, V, W) = \overline{U}$$

$$\mathrm{dist}(X, U) = \left[(X-U)^2 + (Y-V)^2 + (Z-W)^2\right]^{1/2}$$

$$= \frac{2\left[(x-u)^2 + (y-v)^2\right]^{1/2}}{\sqrt{1+x^2+y^2}\,\sqrt{1+u^2+v^2}}$$

$$= \frac{2\,|z-w|}{\sqrt{1+z^2}\,\sqrt{1+w^2}}$$

$$\boxed{\mathrm{dist}(X, U) = \frac{2}{\sqrt{1+z^2}\,\sqrt{1+w^2}}\,|z-w|}$$

Correction factor on length due to distorted geometry

→ Related to a concept in geometry called a Riemannian metric.

→ Measuring distance near equator is most accurate.