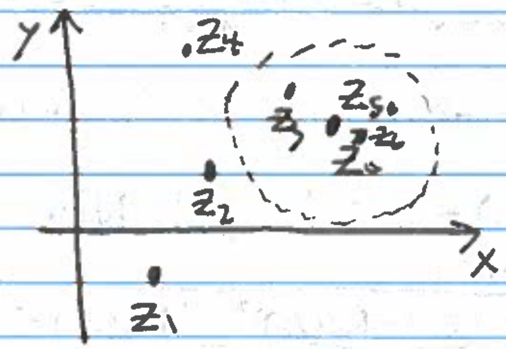
Lecture 9: Limits and Continuity

Definition – A sequence of complex numbers $\{z_n\}$ is said to converge to z_0 and we write

$$\lim_{n\to\infty} z_n = z_0$$

if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|z_n - z_0| < \varepsilon$$



Sequence of real numbers

Theorem – $\lim_{n\to\infty} z_n = z_0 \iff \lim_{n\to\infty} |z_n - z_0| = 0$

~~proof~~:

Let $a_n = |z_n - z_0|$ which is a sequence of positive real numbers. Therefore, $\lim_{n\to\infty} a_n = 0$ if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n| = |a_n| < \varepsilon$.

Squeeze Theorem – If a_n, b_n are real valued sequences satisfying

$$0 \leq a_n \leq b_n$$

and $\lim_{n\to\infty} b_n = 0$ the $\lim_{n\to\infty} a_n = 0$.

Examples

a.) $\lim_{n\to\infty} \left(\frac{i}{3}\right)^n = 0$.

~~proof~~:

Let $a_n = \left|\left(\frac{i}{3}\right)^n - 0\right| = \left|\left(\frac{i}{3}\right)^n\right| = \frac{1}{3^n}$.

Therefore, $\lim_{n\to\infty} a_n = 0$.

B.) $z_n = \frac{2+in}{1+3n}$, $\lim\limits_{n\to\infty} z_n = \frac{i}{3}$

proof:

$$a_n = \left|\frac{2+in}{1+3n} - \frac{i}{3}\right| = \left|\frac{6+3in-i-3in}{3(1+3n)}\right| = \left|\frac{6-i}{3(1+3n)}\right|$$

$$= |6-i|\cdot\frac{1}{3(1+3n)} \le \frac{6}{3(1+3n)} + \frac{1}{3(1+3n)}$$

$$\lim_{n\to\infty} b_n = 0 \Rightarrow \lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} z_n = i/3.$$

C.) If $\lim\limits_{n\to\infty} z_n = z_0 = x_0 + iy_0$ then $\lim\limits_{n\to\infty} \operatorname{Im}(z_n) = y_0$.

proof:

$a_n = |\operatorname{Im}(z_n) - y_0| = |y_n - y_0|$. Now, if we let $b_n = |z_n - z_0|$ then

$$b_n = \sqrt{(\operatorname{Re}(z_n) - \operatorname{Re}(z_0))^2 + (\operatorname{Im}(z_n) - \operatorname{Im}(z_0))^2}$$
$$\ge \sqrt{(\operatorname{Im}(z_n) - \operatorname{Im}(z_0))^2}$$
$$= |\operatorname{Im}(z_n) - \operatorname{Im}(z_0)|$$
$$= a_n.$$

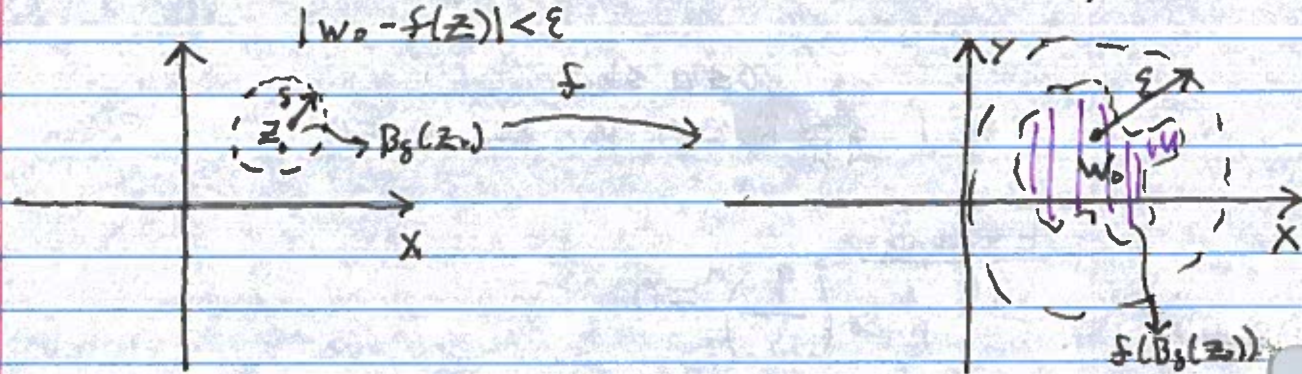
Therefore, since $\lim\limits_{n\to\infty} b_n = 0$ we have that $\lim\limits_{n\to\infty} a_n = 0$ and thus $\lim\limits_{n\to\infty} \operatorname{Im}(z_n) = y_0$.

<u>Definition</u> - Let f be defined on a domain D. We say

$$\lim_{z\to z_0} f(z) = w_0$$

if for all $\varepsilon > 0$ there exists $\delta > 0$ such $|z - z_0| < \delta$ implies

$$|w_0 - f(z)| < \varepsilon$$

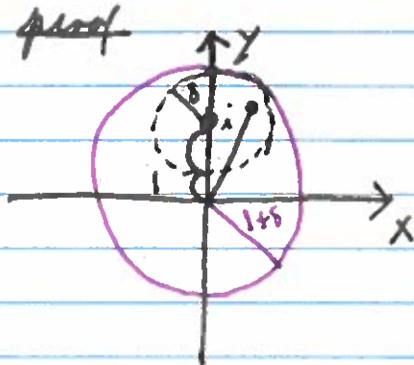


<u>Definition</u> - Let f be a function on a domain D. Then f is continuous at z_0 if $\lim\limits_{z\to z_0} f(z) = f(z_0)$.

Example:

$$\lim_{z \to i} z^2 = -1$$

proof



Let $f(z) = z^2 + 1$. Therefore, we want to show that if $|z - i|$ is small then $|f(z)|$ is small. Now,

$$f(z) = z^2 + 1 = (z+i)(z-i)$$

$$\Rightarrow |f(z)| \leq |(z+i)| \cdot |z-i|$$

$$\leq (|z|+1) \cdot |z-i|$$

$$\leq (1+2\delta) \cdot \delta$$

As $\delta \to 0$ $|f(z)| \to 0$ which proves that

$$\lim_{z \to i} z^2 = -1.$$

Theorem - $\lim_{z \to z_0} f(z) = w_0$ if and only if for all sequences z_n satisfying $\lim_{n \to \infty} z_n = z_0$ implies $\lim_{n \to \infty} f(z_n) = w_0$

Example:

$$\lim_{z \to 0} \frac{Re(z)}{z} \text{ does not exist.}$$

proof:

Let $z_n = i/n$. Then,

$$\frac{Re(z_n)}{z_n} = \frac{0}{i/n} = 0$$

and thus $\lim_{n \to \infty} Re(z_n)/z_n = 0$.

However, if $z_n = 1/n$ we have that

$$\frac{Re(z_n)}{z} = 1/n / 1/n = 1 \not\to 0.$$