MTH 351/651
Homework #3

Due Date: September 16, 2022

1 Problems for Everyone

1. Consider the following dynamical system

$$\dot{x} = ax - x^3,$$

where a is a real number that can be positive, negative, or zero.

   (a) For all three cases find the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions.

   (b) For all three cases, calculate and plot the potential function $V(x)$.

2. The simplest model of malaria assumes that the mosquito population is at equilibrium and models the number of humans infected by malaria, $I(t)$, by the following differential equation:

$$\dot{I} = \frac{\alpha\mu I}{\alpha I + r}(N - I) - \beta I,$$

where $N > 0$ is the total population of healthy and infected individuals and is assumed constant and α, β, r, μ are positive parameters.

   (a) Determine the units of α, β, r and μ.

   (b) Show that there exists a dimensionless change of variables in I and τ such that this system is equivalent to the following dimensionless system

$$\frac{dx}{d\tau} = \frac{Ax}{x + B}(1 - x) - x,$$

   where $A, B > 0$ are dimensionless constants.

   (c) In this dimensionless system the rate of infection is given by

$$f(x) = \frac{Ax}{x + B}.$$

   Calculate $\lim_{x\to\infty} f(x)$, $f'(0)$, and sketch a generic graph of $f(I)$. What effect does changing the parameters A and B have on the graph?

   (d) Calculate the fixed points for this system and analyze there stability.

   (e) Determine the threshold criteria for this model to have an endemic equilibrium.

3. For each of the following problems sketch all qualitatively different phase portraits that occur as r is varied. Sketch a bifurcation diagram of fixed points x^* versus r. In each bifurcation diagram determine what type of bifurcation occurs.

   (a) $\dot{x} = 1 + rx + x^2$.

   (b) $\dot{x} = rx + x^2$.

   (c) $\dot{x} = r - \cosh(x)$.

   (d) $\dot{x} = x - rx(1 - x)$.

   (e) $\dot{x} = x + \dfrac{rx}{1 + x^2}$.

   (f) $\dot{x} = r - 3x^2$.

   (g) $\dot{x} = rx - \dfrac{x}{1 + x^2}$.

   (h) $\dot{x} = rx + \dfrac{x^3}{1 + x^2}$.

4. Consider the system $\dot{x} = rx + x^3 - x^5$, where $r \in \mathbb{R}$ is a parameter.

   (a) Find algebraic expression for all of the fixed points as r is varied.

   (b) Sketch all possible phase portraits as r is varied.

   (c) Sketch the bifurcation diagram for this problem. In this bifurcation diagram, determine what types of bifurcations occur.

   (d) Calculate the explicit values for all of the bifurcation points in this problem.

5. Consider a gene that is activated by the presence of a biochemical substance S. Let $g(t)$ denote the concentration of the gene product at time t, and assume that the concentration of S, denoted by s_0, is fixed. A model describing the dynamics of g is as follows:

$$\frac{dg}{dt} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2},$$

   where the k's are positive constants.

   (a) Determine the units of each of the k's.

   (b) Show that this equation can be put in the dimensionless form

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1 + x^2},$$

   where $r > 0$ and $s \geq 0$ are dimensionless constants.

   (c) In the case when $s = 0$, sketch a bifurcation diagram in the parameter r.

   (d) In the case when $r = .4$, qualitatively sketch a bifurcation diagram in s. Hint, plot $\frac{dx}{d\tau}$ vs x for $s = 0$ and $r = .4$ and think about how this plot qualitatively changes as s is varied.

   (e) Assume that initially there is no gene product, that is $x(0) = 0$, $s = 0$, $r = .4$, and s is slowly increased from zero, i.e. the biochemical substance s is slowly introduced. What happens to $x(\tau)$? Why? What happens if s goes back to zero? Does the gene turn off again?

Homework #3

#1.

Consider the following dynamical system

$$\dot{x} = ax - x^3$$

where a is a real number that can be positive, negative or zero.

(a) For all three cases find the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions.

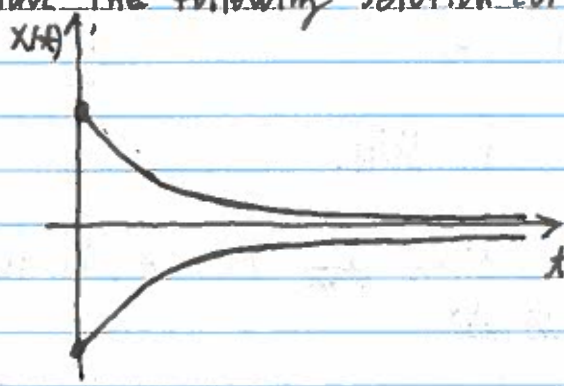
(b) For all three cases, calculate and plot the potential function.

Solution:

(a) The fixed points satisfy $x^* = 0$, $x^* = \pm\sqrt{a}$. Since $\lim_{x\to\infty} ax - x^3 = -\infty$ we have the following cases:

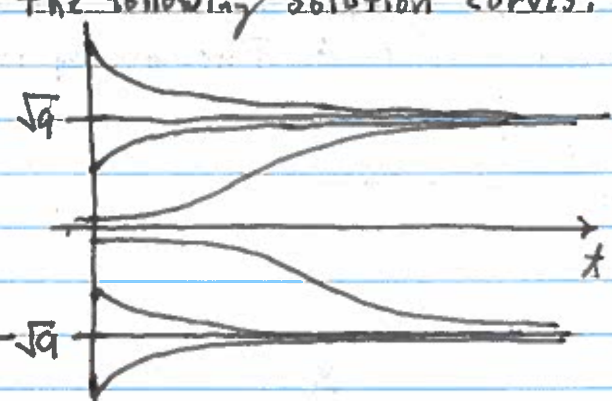
Case 1: $(a \le 0)$

$x^* = 0$ is the only fixed point and it is stable. Thus, we have the following solution curves:



Case 2: $(a > 0)$

$x^* = \pm\sqrt{a}$ are stable while $x = 0$ is unstable. Thus, we have the following solution curves:



#2.

$$\dot{I} = \frac{\alpha \nu I}{\alpha I + r}(N - I) - \beta I$$

Solution:

(a) $[\beta] = 1/\text{time}$

$[r/\alpha] = \text{pop.}$

$[\nu] = 1/\text{time}$

You cannot deduce the units of r and α seperately. However if $[r] = 1/\text{time}$ then $[\alpha] = \text{pop}/\text{time}$ for instance.

(b) Let $x = I/N$ and $\tau = \beta t$. Then

$$\frac{dx}{d\tau} = \frac{dt}{d\tau}\frac{dx}{dt} = \frac{1}{N\beta}\dot{I}$$

$$\Rightarrow \frac{dx}{d\tau} = \frac{\nu N x}{Nx + r/\alpha}\,\frac{N}{N\beta}(1-x) - x,$$

$$= \frac{\nu/\beta\, x}{x + r/\alpha N}(1-x) - x,$$

$$= \frac{Ax}{x+B}(1-x) - x,$$

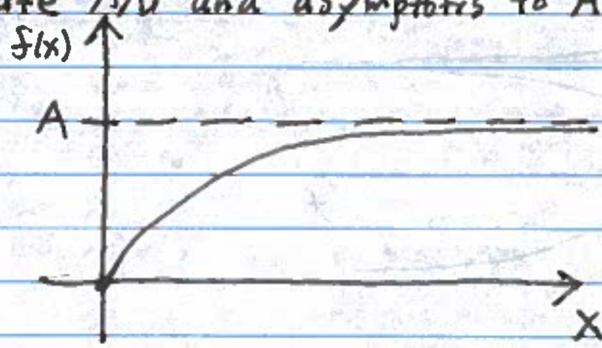
where $A = \nu/\beta$ and $B = r/\alpha N$.

(c) If $f(x) = Ax/x+B$ then $\lim_{x\to\infty} f(x) = A$. Furthermore,

$$f'(x) = \frac{(x+B)A - Ax}{(x+B)^2}$$

$$= \frac{AB}{(x+B)^2} \geq 0.$$

Moreover, $f'(0) = A/B$. Therefore, f is monotone increasing at rate A/B and asymptotes to A. The graph is given below:



(d) The fixed points satisfy

$$\frac{Ax}{x+B}(1-x) - x = 0$$

$$\Rightarrow Ax(1-x) - x(x+B) = 0$$

$$\Rightarrow -(A+1)x^2 + (A-B)x = 0$$

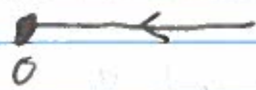
$$\Rightarrow x=0 \text{ or } A-B-(A+1)x = 0$$

$$\Rightarrow x=0 \text{ or } x = \frac{A-B}{A+1}.$$

Since $\lim_{x\to\infty} \frac{Ax}{x+B}(1-x) - x = -\infty$ we have two cases:

Case 1 $(A<B)$:

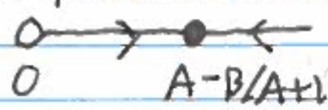
In this case the only non-negative fixed point is $x=0$ and it is stable:



and thus the disease dies out.

Case 2 $(A>B)$:

The phase portrait in this case is given by:



and thus the disease becomes endemic.

From cases 1 and 2 it follows that the threshold condition is $A>B$.

#3

For each of the following problems sketch all qualitatively different phase portraits that occur as r is varied. Sketch a bifurcation diagram of fixed points x^* versus r. In each bifurcation diagram determine what type of bifurcation occurs.

(a) $\dot{x} = 1 + rx + x^2$

(d) $\dot{x} = x - rx(1-x)$

(e) $\dot{x} = x + \dfrac{rx}{1+x^2}$

(g) $\dot{x} = rx - \dfrac{x}{1+x^2}$

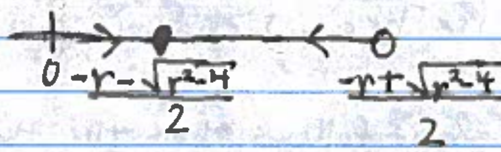
Solution:

(a) $\lim\limits_{x \to \infty} 1 + rx + x^2 = \infty$ and the fixed points satisfy

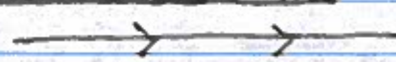
$$x = \frac{-r \pm \sqrt{r^2-4}}{2}$$

and thus we have three cases:

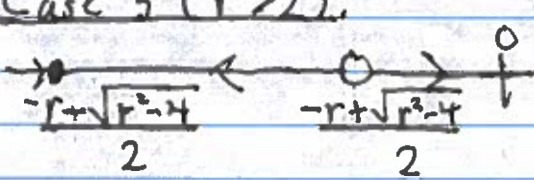
Case 1 $(r<-2)$:



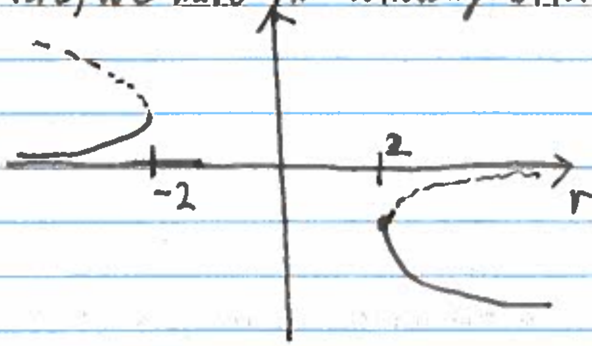
Case 2 $(-2<r<2)$:



Case 3 $(r>2)$:



Therefore, we have the following bifurcation diagram:

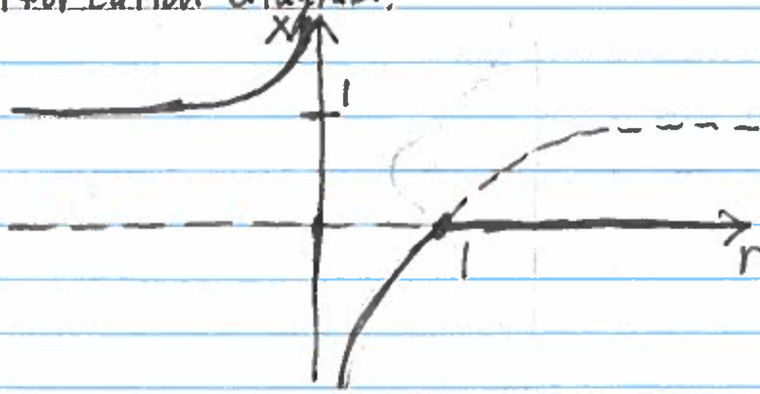


Consequently, this system has two saddle node bifurcations at $r=\pm 2$.

(d) The fixed points satisfy

$$x=0 \quad \text{or} \quad 1-r(1-x)=0$$
$$\Rightarrow x=0 \quad \text{or} \quad 1-r+rx=0$$
$$\Rightarrow x=0 \quad \text{or} \quad x=\frac{r-1}{r}$$

Moreover, $f'(0)=1-r$ and thus $x=0$ is stable if $r>1$ and unstable otherwise. Therefore, we have the following bifurcation diagram:

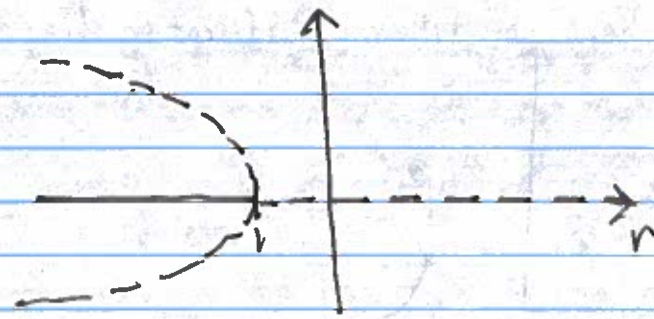


There is a transcritical bifurcation at $r=1$.

(e) The fixed points satisfy

$$x=0 \quad \text{or} \quad 1+x^2+r=0$$
$$\Rightarrow x=0 \quad \text{or} \quad x=\pm\sqrt{-1-r}$$

Since $\lim_{x\to\infty} x+\frac{rx}{1+x^2}=\infty$ it follows that the bifurcation diagram is given by:



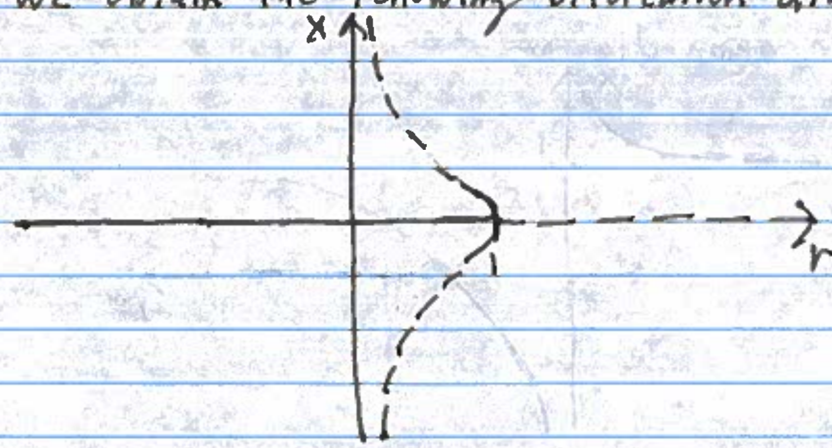
and thus there is a pitchfork bifurcation at $r=-1$.

(g) The fixed points satisfy

$$x=0 \quad \text{or} \quad r+rx^2-1=0$$

$$\Rightarrow x=0 \quad \text{or} \quad x=\pm\sqrt{\frac{1-r}{r}}$$

Moreover, $f'(x)=r-\frac{(1+x^2)-2x^2}{(1+x^2)^2} \Rightarrow f'(0)=r-1$. Consequently, 0 is stable if $r<1$ and is unstable if $r>1$. Therefore, we obtain the following bifurcation diagram.



and thus there is a pitchfork bifurcation at $r=1$. ∎

#4.

Consider the system $\dot{x} = rx + x^3 - x^5$, where $r \in \mathbb{R}$ is a parameter.

(a) Find algebraic expressions for all of the fixed points as r is varied.

(b) Sketch all possible phase portraits as r is varied.

(c) Sketch bifurcation diagram and determine what type of bifurcations occur.

(d) Calculate the explicit values for all of the bifurcation points in this problem.

Solution:

(a) The fixed points satisfy

$$x = 0 \quad \text{or} \quad r + x^2 - x^4 = 0$$

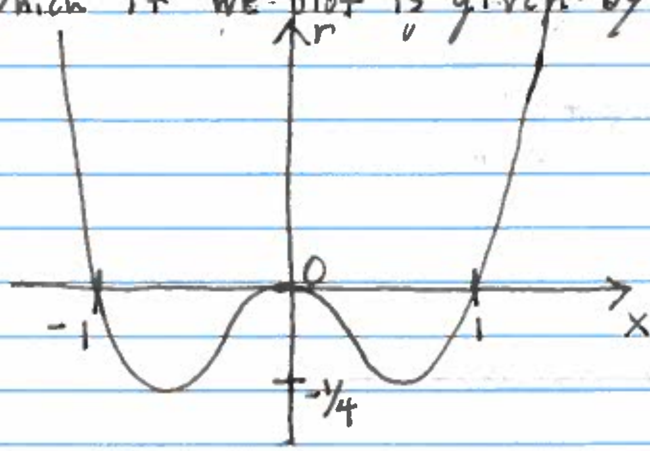
$$\Rightarrow x = 0 \quad \text{or} \quad x^2 = \frac{-1 \pm \sqrt{1+4r}}{-2}$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = \pm\sqrt{\frac{1 \pm \sqrt{1+4r}}{2}}$$

If we solve for r instead we obtain

$$r = x^4 - x^2 = x^2(x^2-1) = x^2(x-1)(x+1).$$

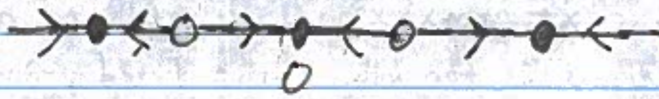
Which if we plot is given by



(b) Since $f'(0) = r$ we obtain 0 is stable if $r < 0$ and unstable if $r > 0$. Therefore, we have the following cases.

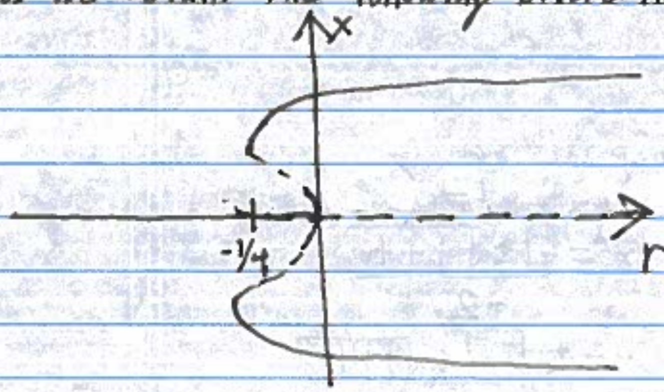
Case 1 ($r < -1/4$):

Case 2 ($-1/4 < r < 0$):



Case 3 ($r > 0$):

(c-d)
Thus we obtain the following bifurcation diagram.



Bifurcation points occur at $r = -1/4$ (two saddle node bifurcations) and $r = 0$ (pithfork).

#5

$$\frac{dq}{dt} = k_1 s_0 - k_2 q + \frac{k_3 q^2}{k_4^2 + q^2}$$

(a) $[k_1] = \text{time}^{-1}$
$[k_2] = \text{time}^{-1}$
$[k_3] = \text{concentration}/\text{time}$.

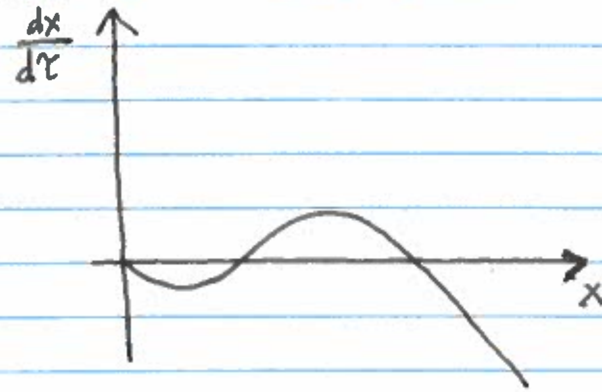
(b) Let $x = q/k_4$ and $\tau = k_3/k_4\, t$. Therefore,

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{k_4}{k_3}\frac{dx}{dt} = \frac{1}{k_3}\frac{dq}{dt} = \frac{k_1}{k_3}s_0 - \frac{k_2 k_4}{k_3}x + \frac{x^2}{1+x^2}$$

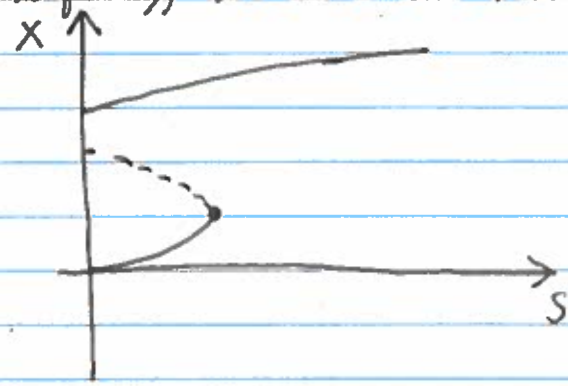
$$\Rightarrow \frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2},$$

where $s = k_1 s_0/k_3$, $r = k_2 k_4/k_3$.

(d) In the case when $r=.4$ and $s=0$ we have the graph



Consequently, we have the following bifurcation diagram



(e) If s is raised above the bifurcation point the gene concentration shoots to the higher solution branch. If s is then lowered, solutions remain on the upper branch. This process is illustrated in the following hysterisis diagram.