MTH 351/651
Homework #4

Due Date: September 30, 2022

1 Problems for Everyone

1. For each of the following dynamical systems on the circle S^1, find and classify all the fixed points, and sketch the phase portrait on the circle.

   (a) $\dot{\theta} = 1 + 2\cos(\theta)$

   (b) $\dot{\theta} = \sin^3(\theta)$

   (c) $\dot{\theta} = 3 + \cos(2\theta)$

   (d) $\dot{\theta} = \sin(\theta) + \cos(\theta)$

2. At 12:00, the hour and minute hands of a clock are perfectly aligned. When is the next time they will be aligned?

3. For each of the following dynamical systems on the circle S^1, sketch all of the qualitatively different phase portraits on S^1 that occur as μ is varied. Classify the bifurcations that occur as μ varies and find all the bifurcation values μ. **Hint:** Do not forget that in addition to a zero, the sign of a function can change across a vertical asymptote.

   (a) $\dot{\theta} = \mu + \cos(\theta) + \cos(2\theta)$,

   (b) $\dot{\theta} = \dfrac{\sin(\theta)}{\mu + \sin(\theta)}$.

   (c) $\dot{\theta} = \dfrac{\sin(2\theta)}{1 + \mu\sin(\theta)}$.

Homework #4

#1.

For each of the following dynamical systems on S^1, find and classify all the fixed points, and sketch the phase portrait on the circle.

(a) $\dot{\theta} = 1 + 2\cos\theta$

(b) $\dot{\theta} = \sin^3\theta$

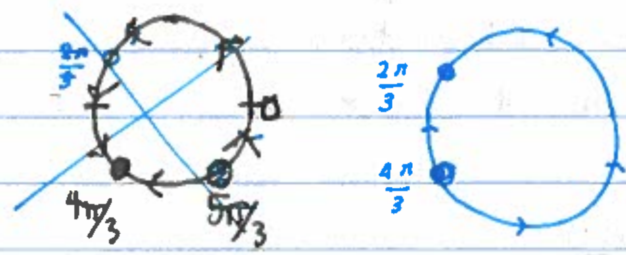
(c) $\dot{\theta} = 3 + \cos(2\theta)$

(d) $\dot{\theta} = \sin\theta + \cos\theta$

Solution:

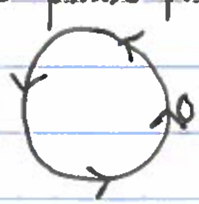
(a) The fixed points are given by $\theta_1 = 4\pi/3$ and $\theta_2 = \cancel{5\pi/3}$. $2\pi/3$

Since $\dot{\theta}|_{\theta=0} = 3$ we obtain the following phase portrait:

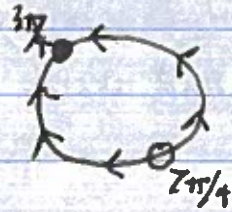


(b) The fixed points are given by $\theta = 0$ and $\theta = \pi$. Since $\dot{\theta}|_{\theta=\pi/2} = 1$ we obtain the following phase portrait:

(c) Since $\dot{\theta} > 0$ for all t there are no fixed points and thus the phase portrait is given by



(d) The fixed points satisfy $\tan\theta = -1$ and are thus given by $\theta = 3\pi/4$ and $\theta = 7\pi/4$. Consequently, since $\dot{\theta}|_{\theta=0} = 1$ we obtain the following phase portrait:



#2

At 12:00, the hour and minute hands of a clock are perfectly aligned. When is the next time they will be aligned.

Solution:

If we let θ, ϕ denote the angular coordinates of the hour and minute hands and t denote minutes it follows that

$$\theta = \frac{2\pi}{12\cdot 60} t \quad \text{and} \quad \phi = \frac{2\pi}{60} t$$

$$\Rightarrow \phi - \theta = \frac{2\pi}{60}\left(\frac{11}{12}\right) t$$

Setting the phase difference equal to 2π we obtain

$$\frac{2\pi}{60}\left(\frac{11}{12}\right) t = 2\pi$$

$$\Rightarrow t = \frac{60\cdot 12}{11} \text{ minutes}$$

$$= 65.\overline{45} \text{ minutes}$$

and thus the hands overlap at approximately 1:09.

#3.

For each of the following dynamical systems on S^1, sketch all qualitatively different phase portraits that occur as μ is varied. Classify the bifurcations that occur as μ varies and find the bifurcation values μ.

(a) $\dot{\theta} = \mu + \cos\theta + \cos(2\theta)$

(b) $\dot{\theta} = \dfrac{\sin\theta}{\mu + \sin\theta}$

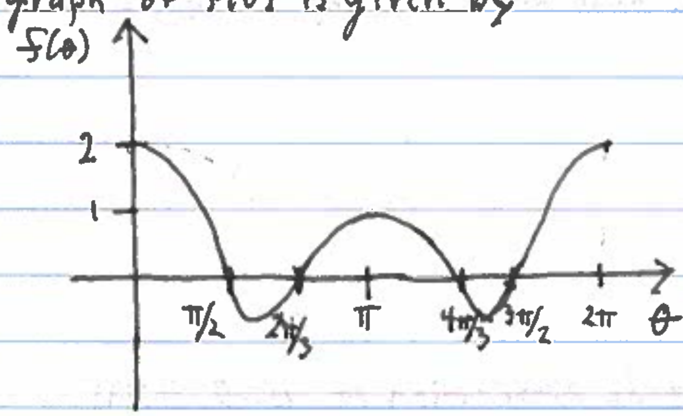
(c) $\dot{\theta} = \dfrac{\sin(2\theta)}{1 + \mu\sin\theta}$

Solution:

(a) $\dot{\theta} = \mu + \cos\theta + 2\cos^2\theta - 1$

$\quad = \mu - 1 + \cos\theta + 2\cos^2\theta$

If we let $f(\theta) = \cos\theta + 2\cos^2\theta = \cos\theta(1 + 2\cos\theta)$ it follows that the graph of $f(\theta)$ is given by



$f'(\theta) = -\sin\theta - 4\cos\theta\sin\theta$ and thus the critical points of f are located at:

$$\sin\theta(1 + 4\cos\theta) = 0$$

$$\Rightarrow \theta_1^* = 0,\ \theta_2^* = \pi,\ \theta_3^* = 3\pi/4,\ \theta_4^* = 5\pi/4$$

Now,

$$f(\theta_1^*)=2,\ f(\theta_2^*)=1,\ f(\theta_3^*)=1/\sqrt{2}-2,\ f(\theta_4^*)=1/\sqrt{2}-2$$

and thus bifurcations occur when

$$2+\nu-1=0 \Rightarrow \nu=-1$$

$$1+\nu-1=0 \Rightarrow \nu=0$$

$$1/\sqrt{2}+\nu-2-1=0 \Rightarrow \nu=3-1/\sqrt{2}$$

Therefore, we have the following cases

Case 1:

$\nu<-1 \Rightarrow$ No fixed points and flow is in negative direction

Case 2:

$-1<\nu<0 \Rightarrow 2$ fixed points

and thus there is a saddle node bifurcation at $\nu=-1,\ \theta=0$

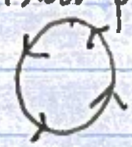
Case 3:

$0<\nu<3-1/\sqrt{2} \Rightarrow 4$ fixed points

and thus there is a saddle node bifurcation at $\nu=0,\ \theta=\pi$

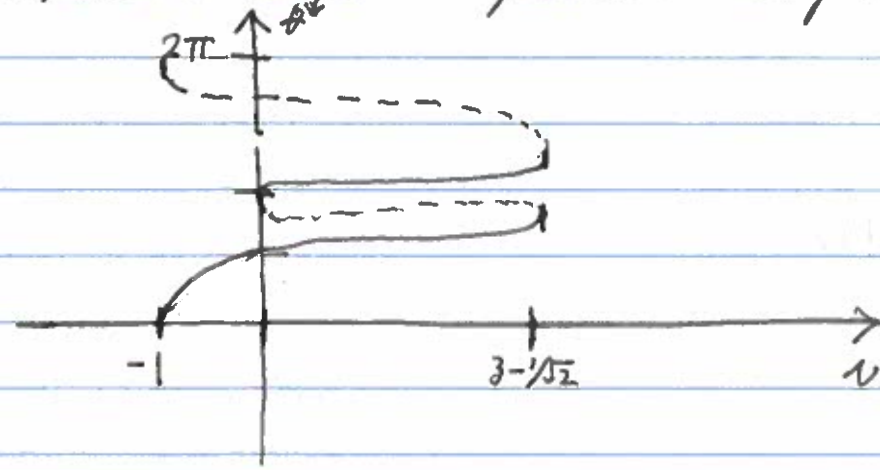
Case 4:

$0<\nu<3-1/\sqrt{2} \Rightarrow$ No fixed points and flow is in positive direction



and thus there are two saddle node bifurcations at $\nu=-1,\ \theta=3\pi/4$ and $\nu=-1,\ \theta=5\pi/$

Therefore, we obtain the following bifurcation diagram



(b) $\dot{\theta} = \frac{\sin\theta}{\nu + \sin\theta}$ has fixed points at $\theta = 0, \pi$. However, when $|\nu| < 1$ this function has asymptotes across which $\dot{\theta}$ changes sign. We also have that

$$\frac{d}{d\theta} \frac{\sin\theta}{\nu+\sin\theta}\Bigg|_{\theta=0} = \frac{(\nu+\sin\theta)\cos\theta - \sin\theta\cos\theta}{(\nu+\sin\theta)^2}\Bigg|_{\theta=0} = \frac{1}{\nu}.$$

Therefore, we have the following cases

Case 1 ($\nu < -1$):

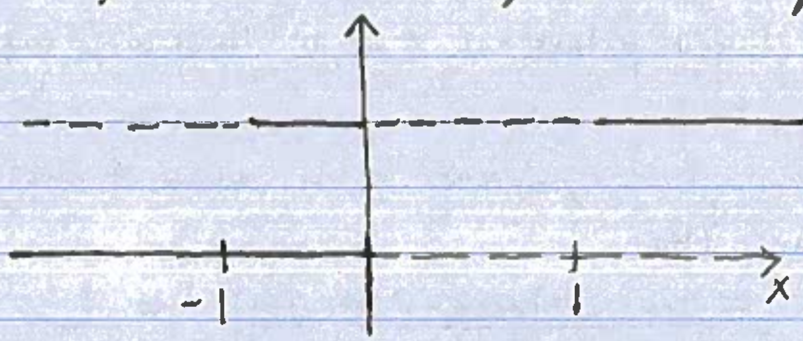
Case 2 ($-1 < \nu < 0$):

Where the X indicates a vertical asymptote.

Case 3 $(0<\nu<1)$:

Case 4 $(\nu \geq 1)$:

Therefore, we have the following bifurcation diagram.



(c) $\dot{\theta} = \frac{\sin(2\theta)}{1+\nu\sin\theta}$ has fixed points at $\theta = 0, \pi/2, \pi, 3\pi/2$. However, when $|\nu|>1$ the function also has two asymptotes across which the function changes sign. Furthermore,

$$\frac{d\dot{\theta}}{d\theta} = \frac{(1+\nu\sin\theta)2\cos(2\theta) - \nu\cos(\theta)\sin(2\theta)}{(1+\nu\sin\theta)^2}$$

$$\Rightarrow \left.\frac{d\dot{\theta}}{d\theta}\right|_0 = 2$$

$$\left.\frac{d\dot{\theta}}{d\theta}\right|_{\pi/2} = \frac{(1+\nu)(-2)}{(1+\nu)^2} = \frac{-2}{1+\nu}$$

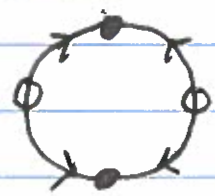
$$\left.\frac{d\dot{\theta}}{d\theta}\right|_{\pi} = 2$$

$$\left.\frac{d\dot{\theta}}{d\theta}\right|_{3\pi/2} = \frac{(1-\nu)2}{(1-\nu)^2} = \frac{-2}{1-\nu}.$$

Therefore, we have the following cases

Case 1 ($\nu<-1$):

Case 2 ($-1<\nu<1$):



Case 3 ($\nu>1$):

Therefore, we have the following bifurcation diagram.