MTH 351/651

Homework #5

Due Date: October 07, 2022

1 Problems for Everyone

2 1. Consider the arms race described by

$$\dot{x} = y,$$
$$\dot{y} = -x + y.$$

   (a) How would you describe the interaction between these two nations in practical terms?
   (b) Classify the fixed point at the origin. What does this imply about the arms race?
   (c) Sketch $x(t)$ and $y(t)$ as functions of t, assuming $x(0) = 1$, $y(0) = 0$.

3 2. In each of the following, predict the course of the following arms races depending on the relative sizes of a and b (assume $a, b > 0$). Explain in practical terms how the countries interact, and try to think of examples of countries that might interact in this manner.

   (a) $\dot{x} = ay$ and $\dot{y} = bx$
   (b) $\dot{x} = ax + by$ and $\dot{y} = -bx - ay$
   (c) $\dot{x} = ax + by$ and $\dot{y} = bx + ay$
   (d) $\dot{x} = 0$ and $\dot{y} = ax + by$

3 3. Here are the official definitions of the various types of stability. Consider a fixed point $\mathbf{x}^*$ of a system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{F} : \mathbb{R}^n \mapsto \mathbb{R}^n$.

   - We say $\mathbf{x}^*$ is **attracting** if there exists a $\delta > 0$ such that if $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$ then $\lim_{t \to \infty} \mathbf{x}(t) = \mathbf{x}^*$.
   - We say $\mathbf{x}^*$ is **Liapunov stable** if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $t \geq 0$ if $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$ then $\|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon$.

   For each of the following systems, determine whether the origin is attracting, Liapunov stable, asymptotically stable, or none of the above.

   (a) $\dot{x} = y$ and $\dot{y} = -4x$
   (b) $\dot{x} = 0$ and $\dot{y} = -y$
   (c) $\dot{x} = -x$ and $\dot{y} = -5y$
   (d) $\dot{x} = x$ and $\dot{y} = y$

2 4. For a 2×2 matrix A prove that the eigenvalue of A satisfy

$$\lambda_{1.2} = \frac{\mathrm{Tr}(A)}{2} \frac{\pm 1}{2} \sqrt{(\mathrm{Tr}(A))^2 - 4\det(A)}.$$

Homework #5

#1.

Consider the arms race described by

$$\dot{x} = y$$
$$\dot{y} = -x + y$$

(a) How would you describe the interaction between these two nations in practical terms?

(b) Classify the fixed point at the origin. What does this imply about the arms race.

(c) Sketch $x(t)$ and $y(t)$ as a function of t, assuming $x(0)=1$, $y(0)=0$.

Solution:

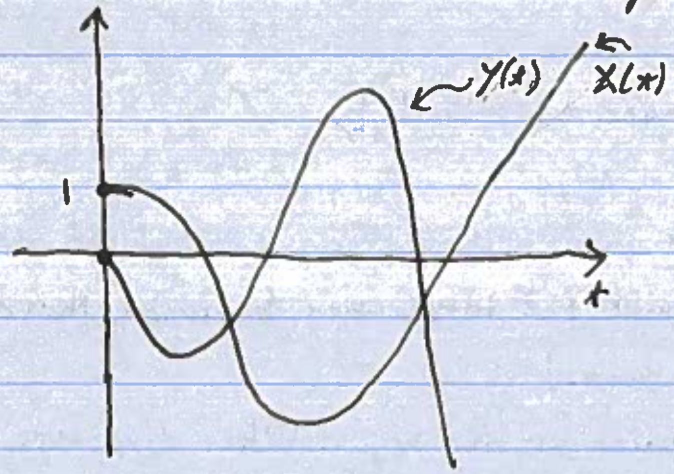
(a) Country x fears country y and increases its arms in response to a buildup of y's military. Country y is an arms dealer and will decrease its arms in response to a buildup of y probably out of fear as well. This could describe a relationship such as Libya (country y) and Israel (country x) in the 1980's.

(b) The matrix is $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ which has eigenvalues satisfying

$$\lambda_1 + \lambda_2 = 1 \text{ and } \lambda_1 \lambda_2 = 1$$
$$\Rightarrow \lambda_1^2 - \lambda_1 + 1 = 0$$
$$\Rightarrow \lambda_1 = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

Consequently, the fixed point is an unstable spiral and thus the countries will uncontrollably spiral between build of arms and disarmament.

(c) The solutions will behave something like



#2

In each of the following predict the course of the following arms races depending on the relative sizes of a and b (assume $a,b>0$). Explain in practical terms how the countries interact, and try to think of examples of countries that might interact in this manner.

(a) $\dot{x}=ay$ and $\dot{y}=bx$

(b) $\dot{x}=ax+by$ and $\dot{y}=-bx-ay$

(c) $\dot{x}=ax+by$ and $\dot{y}=bx+ay$

(d) $\dot{x}=0$ and $\dot{y}=ax+by$

Solution:

(a) These two countries mutually distrust each other and compete in both arms and nation building. The eigenvalues satisfy

$$\lambda_1+\lambda_2=0 \quad \text{and} \quad \lambda_1\lambda_2=-ab$$

$$\Rightarrow \lambda_{1,2}=\pm\sqrt{ab}$$

and thus the system leads to an uncontrolled arms race or uncontrolled nation building. This could mimic U.S.-China relations.

(b) Country x is an arms dealer that fears country y. Country y wants a balance between military spending and infrastructure but at time same time wants to exploit weaknesses in country x's military. The eigenvalues satisfy

$$\lambda_1 + \lambda_2 = 0 \,, \cancel{\lambda_1 \lambda_2 = b^2}$$

$$\Rightarrow \lambda_{1,2} = \cancel{\pm bi} . \quad \pm\sqrt{a^2 - b^2}$$

if $b < a$

Consequently, both countries will continually oscillate between building up arms and nation building.

(c) Countries x and y are arms dealers and build up weepons in response to each other's arms. The eigenvalues satisfy

$$\lambda_1 + \lambda_2 = 2a \,, \quad \lambda_1 \lambda_2 = a^2 - b^2$$

$$\Rightarrow (2a - \lambda_2)\lambda_2 = a^2 - b^2$$

$$\Rightarrow \lambda^2 - 2a\lambda_2 + a^2 - b^2 = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{2a \pm \sqrt{4a^2 - 4a^2 + 4b^2}}{2}$$

$$= a \pm b.$$

Consequently, the fixed point is an unstable node if $a > b$ and is a saddle if $a < b$.

#3.

For each of the following systems determine whether the origin is attracting, Liapunov stable, asymptotically stable, or none of the above.

(a) $\dot{x} = y$ and $\dot{y} = -4x$

(b) $\dot{x} = 0$ and $\dot{y} = -y$

(c) $\dot{x} = -x$ and $\dot{y} = -5y$

(d) $\dot{x} = x$ and $\dot{y} = y$

Solution:

(a) The eigenvalues are given by $\lambda_1, \lambda_2 = 2i$ and thus the origin is Liapunov stable.

(b) The line $y = 0$ is a stable line of fixed points and thus the origin is Liapunov stable.

(c) The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -5$ and thus the origin is asymptotically stable.

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#4.

For a 2x2 matrix A prove that the eigenvalues of A satisfy

$$\lambda_{1,2} = \frac{Tr(A) \pm \sqrt{Tr(A)^2 - 4det(A)}}{2}.$$

proof:

Since $Tr(A) = \lambda_1 + \lambda_2$ and $det(A) = \lambda_1 \lambda_2$ we have that

$$det(A) = \lambda_1 (Tr(A) - \lambda_1)$$

$$\Rightarrow \lambda_1^2 - Tr(A)\lambda_1 + det(A) = 0$$

$$\Rightarrow \lambda_1 = \frac{Tr(A) \pm \sqrt{Tr(A)^2 - 4det(A)}}{2}.$$

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