# MTH 351/651
# Homework #6

Due Date: October 12, 2022

## 1 Problems for Everyone

4 1. Consider the linear system

$$\dot{x} = x - y,$$
$$\dot{y} = x + y.$$

(a) Show that the system has eigenvalues $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$ with eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

(b) Express the generic solution

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

as a sum of two real-valued functions. **Hint:** First, express $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Second, use the fact that for $\omega \in \mathbb{R}$, $e^{i\omega} = \cos(\omega) + i\sin(\omega)$ to rewrite the solution $\mathbf{x}(t)$ in terms of sines and cosines and then separate the terms that have a prefactor of $i$ from those that do not. Third, find conditions on $a_1, a_2, b_1, b_2$ that ensure the terms involving $i$ vanish.

2. For the following systems, find the nullclines and the fixed points and plot a plausible phase portrait for the system. You do not have to analyze the stability of the fixed points.

2 (a) $\dot{x} = x - x^3$ and $\dot{y} = -y$.

2 (b) $\dot{x} = y$ and $\dot{y} = x(1 + y) - 1$.

(c) $\dot{x} = \sin(y)$ and $\dot{y} = x - x^3$.

2 (d) $\dot{x} = xy - 1$ and $\dot{y} = x - y^3$.

Homework #6

#1

Consider the linear system

$$\dot{x} = x - y$$
$$\dot{y} = x + y$$

(a) Show that the system has eigenvalues $\lambda_1 = 1+i$, $\lambda_2 = 1-i$ with eigenvectors

$$\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

(b) Express the generic solution

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

as a sum of two real valued functions.

Solution:

(a) The system can be expressed in the form

$$\dot{\vec{x}} = A\vec{x},$$

where $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ which has eigenvalues satisfying

$$\lambda_1 + \lambda_2 = 2 \quad \lambda_1\lambda_2 = 2$$
$$\Rightarrow \lambda_1(2-\lambda_1) = 2$$
$$\Rightarrow \lambda_1^2 - 2\lambda_1 + 2 = 0$$
$$\Rightarrow \lambda_{1,2} = 1 \pm i.$$

Since $A - \lambda_1 I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$ it follows that the two eigenvectors are given by

$$\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \overline{\vec{v}_1} = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

(b) The generic solution to the ODE is therefore,

$$\vec{x}(t) = (a_1+ib_1)e^{(1+i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} + (a_2+ib_2)e^{(1-i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

$$= e^t\left((a_1+ib_1)e^{it}\begin{bmatrix} i \\ 1 \end{bmatrix} + (a_2+ib_2)e^{-it}\begin{bmatrix} -i \\ 1 \end{bmatrix}\right)$$

$$= e^t\begin{bmatrix} i(a_1+ib_1)(\cos(t)+i\sin(t)) - i(a_2+ib_2)(\cos(t)-i\sin(t)) \\ (a_1+ib_1)(\cos(t)+i\sin(t)) + (a_2+ib_2))(\cos(t)-i\sin(t) \end{bmatrix}$$

If we choose $a_1 = a_2$ and $b_2 = -b_1$ we have that

$$\vec{x}(t) = e^t\begin{bmatrix} i(a_1+ib_1)(\cos(t)+i\sin(t)) - i(a_1-ib_1)(\cos(t)-i\sin(t)) \\ (a_1+ib_1)(\cos(t)+i\sin(t)) + (a_1-ib_1)(\cos(t)-i\sin(t)) \end{bmatrix}$$

$$= e^t\begin{bmatrix} -2a_1\sin(t) - 2b_1\overset{\cos}{\cancel{\sin}}(t) \\ 2a_1\cos(t) - 2b_1\sin(t) \end{bmatrix}$$

∎

## #2

For the following systems, find the nullclines and fixed points, and plot a reasonable phase portrait.

(a) $\dot{x} = x - x^3$, $\dot{y} = -y$

(b) $\dot{x} = y$, $\dot{y} = x(1+y) - 1$

(c) $\dot{x} = \sin(y)$, $\dot{y} = x - x^3$

(d) $\dot{x} = xy - 1$, $\dot{y} = x - y^3$

**Solution:**

(a) The nullclines are given by

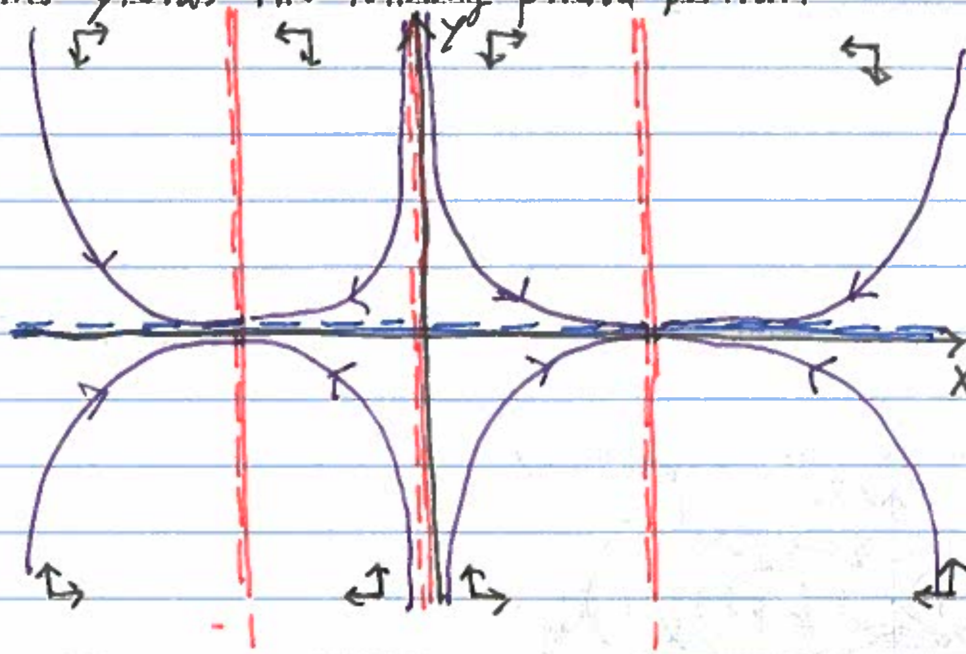
N1 ($\dot{x} = 0$):

$x = 0$, $x = \pm 1$

N2 ($\dot{y} = 0$):

$y = 0$

This yields the following phase portrait




(b) The nullclines are given by

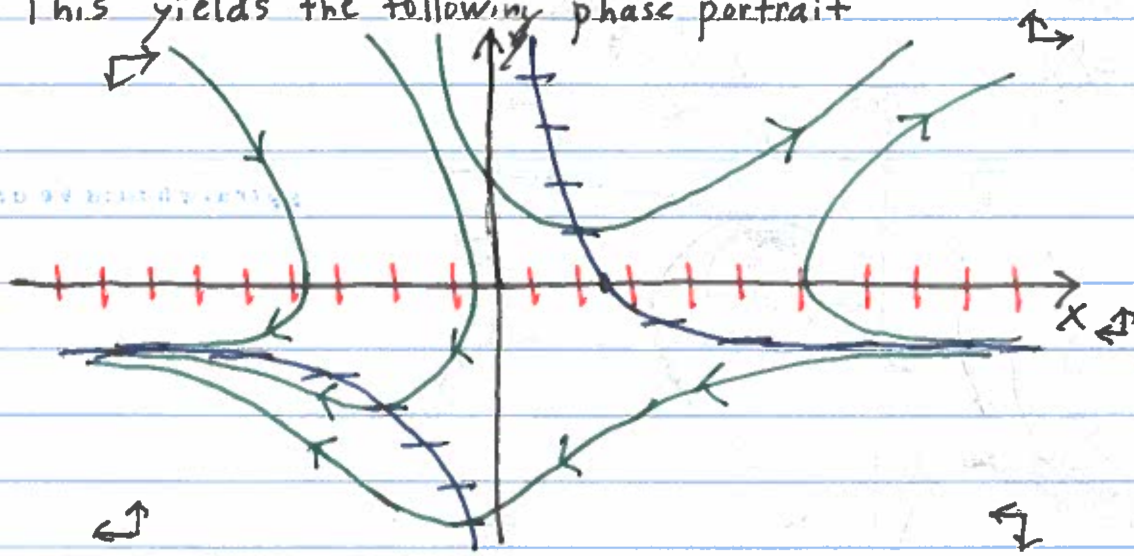
<u>N1 ($\dot{x}=0$):</u>

$y=0$

<u>N2 ($\dot{y}=0$):</u>

$y = 1/x - 1 = \dfrac{1-x}{x}$

This yields the following phase portrait



(c) The nullclines are given by

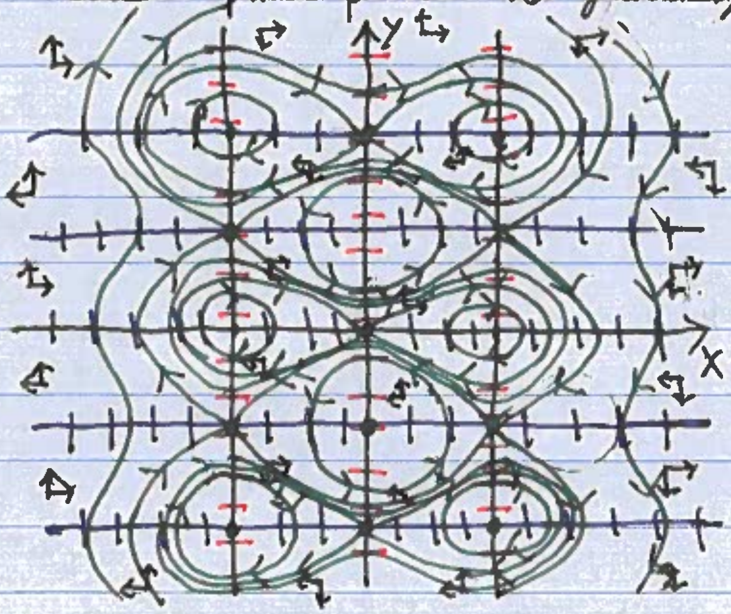
N1 ($\dot{x}=0$):

$y = n\pi$

N2 ($\dot{y}=0$):

$x = 0, \pm 1$

Possible phase portrait is given by:



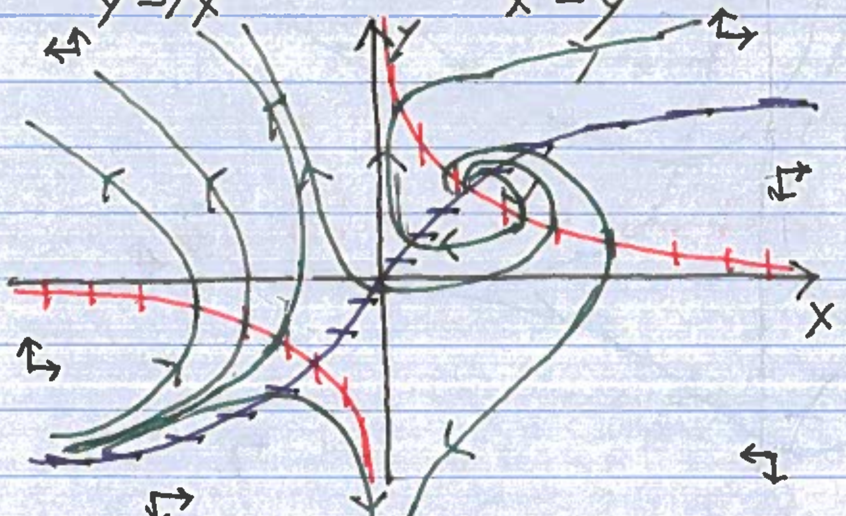
(d) The nullclines are given by

N1 ($\dot{x}=0$):

$y = 1/x$

N2 ($\dot{y}=0$):

$x = y^3$



spiral should be at $(-1,-1)$.