MTH 351/651
Homework #7

Due Date: October 21, 2022

1 Problems for Everyone

1. Consider the system $\dot{x} = y^3 - 4x$, $\dot{y} = y^3 - y - 3x$.
   (a) Find all the fixed points and classify them.
   (b) Show that the line $x = y$ is invariant, i.e. any trajectory that starts on it stays on it.
   (c) Show that $|x(t) - y(t)| \to 0$ as $t \to \infty$ for all trajectories.
   (d) Sketch the phase portrait.
   (e) Use the StreamPlot command in Mathematica to plot an accurate phase portrait on the square domain $-20 \le x, y \le 20$. Notice the trajectories seem to approach a curve as $t \to -\infty$; explain this curve intuitively and find an approximate equation for the curve. You do not have to submit your Mathematica code or any output of your code.

2. In this problem we study the competing species model:
$$\begin{cases} \dot{x} = ax + bxy \\ \dot{y} = cy + dxy \end{cases},$$
   where $a, b, c, d \in \mathbb{R}$. In the following cases, sketch the phase portrait, analyze the stability of any fixed points, and give a biological interpretation for each case.
   (a) $a, b, c, d > 0$.
   (b) $a, b, c > 0$ and $d < 0$.
   (c) $a, b, d > 0$ and $c < 0$.
   (d) $a, b > 0$ and $c, d < 0$.
   (e) $b, c > 0$ and $a, d < 0$.
   (f) $a, c > 0$ and $b, d < 0$.
   (g) $b, d > 0$ and $a, c < 0$.
   (h) $b > 0$ and $a, c, d < 0$.
   (i) $a > 0$ and $b, c, d < 0$.
   (j) $a, b, c, d < 0$.

3. Consider the following model for the interaction of the population of deer N_1 and rabbits N_2:
$$\begin{cases} \dot{N}_1 = r_1 N_1 \left(1 - \frac{N_1}{\kappa_1}\right) - \alpha N_1 N_2 \\ \dot{N}_2 = r_2 N_2 \left(1 - \frac{N_1}{\kappa_2}\right) - \beta N_1 N_2 \end{cases},$$
   where $r_1, r_2, \kappa_1, \kappa_2, \alpha, \beta$ are constants.

(a) Give biological interpretations of each of the parameters.

(b) Nondimensionalize this system. There are many ways to do this. You should do the most natural one that makes sense biologically and makes the problem as simple as possible.

(c) Classify the fixed points for this system and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.

4. Consider the system $\dot{x} = xy$, $\dot{y} = x^2 - y$.

   (a) Show that the linearization predicts the origin is a non-isolated fixed point.

   (b) Show in fact that the origin is an isolated fixed point.

   (c) Sketch the phase portrait for this system.

   (d) Is the origin repelling, attracting, a saddle, or what?

5. Consider the system $\dot{x} = -y - x^3$, $\dot{y} = x$. Prove that the origin is a spiral, although the linearization predicts a center.

6. The system $\dot{x} = xy - x^2y + y^3$, $\dot{y} = y^2 + x^3 - xy^2$ has a fixed point at the origin that is difficult to analyze. Sketch the phase portrait for this system. Converting to polar coordinates could be useful.

Homework #7

#1.

Consider the system $\dot{x} = y^3 - 4x$, $\dot{y} = y^3 - y - 3x$.

(a) Find all the fixed points and classify them.

(b) Show that the line $y = x$ is invariant.

(c) Show that $|x(t) - y(t)| \to 0$ as $t \to \infty$ for all trajectories.

(d) Sketch the phase portrait.

Solution:

(a) Fixed points satisfy

$$y^3 - 4x = 0 \quad \text{and} \quad y^3 - y - 3x = 0$$

$$\Rightarrow x = y^3/4 \quad \text{and} \quad y^3 - y - 3y^3/4 = 0$$

$$\Rightarrow x = y^3/4 \quad \text{and} \quad 4y^3 - 4y - 3y^3 = 0$$

$$\Rightarrow x = y^3/4 \quad \text{and} \quad y = 0, \pm 2$$

Therefore the fixed points are $(0,0)$, $(-2,-2)$, $(2,2)$. The Jacobian is given by

$$J(x,y) = \begin{bmatrix} -4 & 3y^2 \\ -3 & 3y^2 - 1 \end{bmatrix}$$

$$\Rightarrow J(0,0) = \begin{bmatrix} -4 & 0 \\ -3 & -1 \end{bmatrix}, \quad J(2,2) = \begin{bmatrix} -4 & 12 \\ -3 & 11 \end{bmatrix}, \quad J(-2,-2) = \begin{bmatrix} -4 & 12 \\ -3 & 11 \end{bmatrix}$$

Therefore $J(0,0)$ has eigenvalues $\lambda_{1,2} = -4, -1$ and thus $(0,0)$ is a stable node.

For $J(\pm 2, \pm 2)$ the eigenvalues satisfy

$$\lambda_1 + \lambda_2 = 8, \quad \lambda_1 \lambda_2 = -8$$

Consequently, the eigenvalues of are opposite signs and thus $(\pm 2, \pm 2)$ are both saddles.

(b) If we let $z = y - x$ then

$$\dot{z} = \dot{y} - \dot{x} = y^3 - y - 3x - y^3 + 4x = x - y = -z.$$

Therefore, if $z_0 = y_0 - x_0 = 0$ then $\dot{z} = 0$ and thus $y = x$ is invariant.

(c) For all z_0, $\lim_{t\to\infty} z(t) = 0$ since $z = 0$ is a global stable fixed point of $\dot{z} = -z$. Consequently, $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} y(t) - x(t) = 0$.

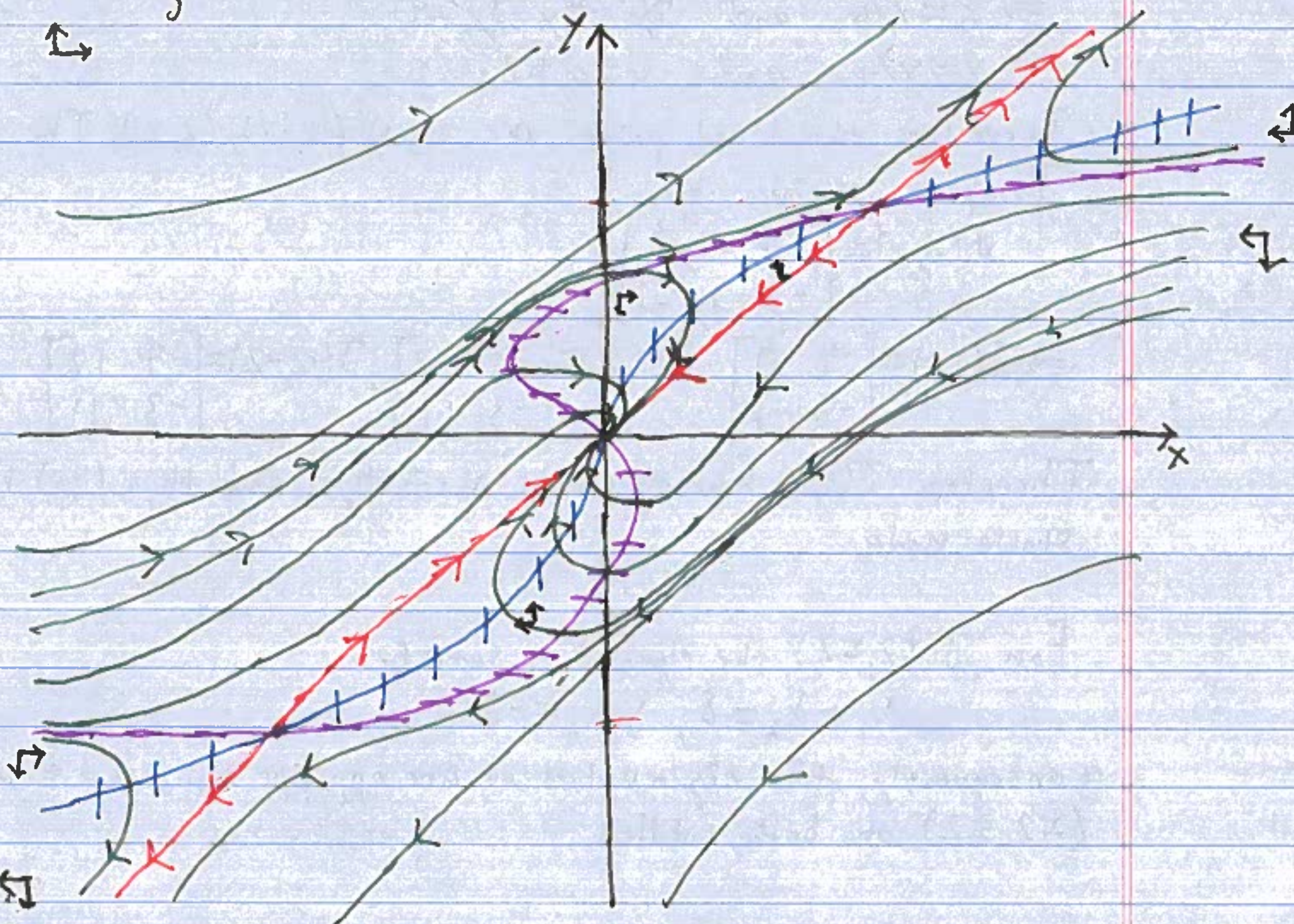
(d) The nullclines are given by

N1 ($\dot{x} = 0$):

$$x = y^3/4$$

N2 ($\dot{y} = 0$):

$$x = \frac{y^3 - y}{3}$$



#2

In this problem we study the competing species model

$$\dot{x} = ax + bxy$$
$$\dot{y} = cy + dxy$$

where $a, b, c, d \in \mathbb{R}$. In the following cases, sketch the phase portrait, analyze the stability of any fixed points, and give a biological interpretation for each case.

(b) $a, b, c > 0,\ d < 0$

(c) $a, b, d > 0,\ c < 0$

(f) $a, c > 0,\ b, d < 0$

(g) $b, d > 0,\ a, c < 0$

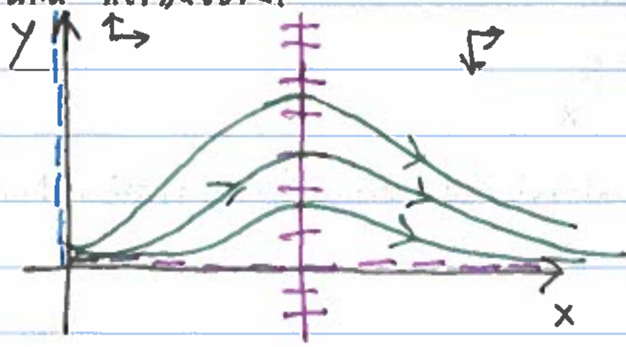
Solution:

For all of these problems the fixed points are $(0,0)$, $(-c/d, -a/b)$. The Jacobian at the fixed points is given by

$$J(0,0) = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}, \quad J(-c/d, -a/b) = \begin{bmatrix} 0 & -cb/d \\ -ad/b & 0 \end{bmatrix}$$

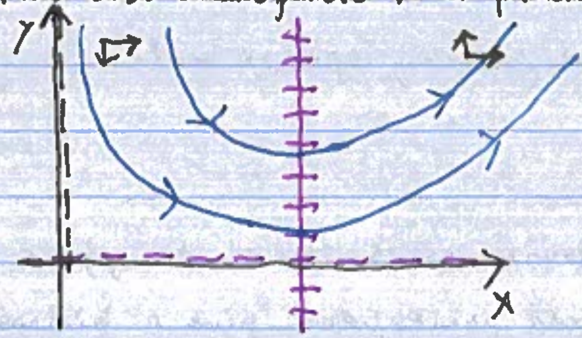
Thus the eigenvalues of $J(0,0)$ are $\lambda_{1,2} = a, c$ while the eigenvalues of $J(-c/d, -a/b)$ are $\lambda_1, \lambda_2 = \pm\sqrt{ac}$. Furthermore, since this system is conservative all of the fixed points are saddles and nonlinear centers.

(b) This case corresponds to the interaction between an omnivore and herbavore.



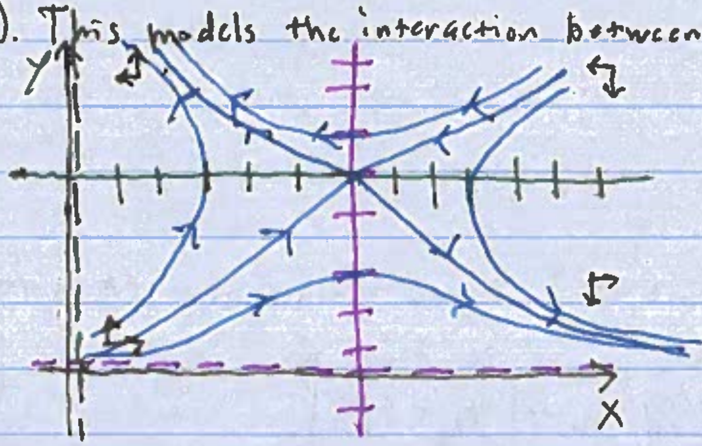
In this case the omnivores dominate.

(c) This case corresponds to a predator and omnivore.



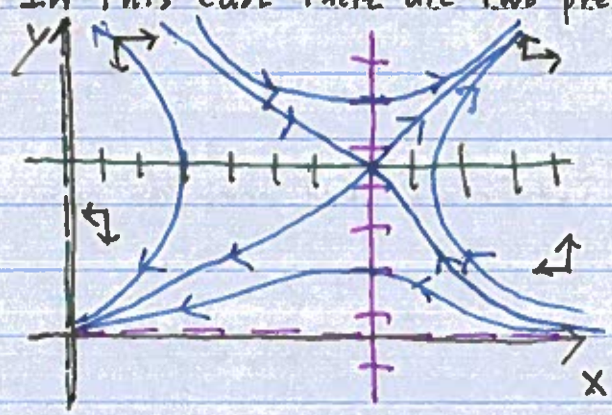
In this case both species thrive.

(f). This models the interaction between two herbavores competing for resources.



In this case, depending on initial conditions, one species dominates over the other.

(g) In this case there are two predators who are consuming each other.



In this case, depending on initial conditions, both species thrive or they both go extinct.

∎

#3

Consider the following model for the interaction of deer N_1 and rabbits N_2:

$$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - \alpha N_1 N_2$$

$$\dot{N}_2 = r_2 N_2 (1 - N_2/K_2) - \beta N_1 N_2$$

(a) Give biological interpretations of each parameter.

(b) Nondimensionalize this system.

(c) Classify fixed points and sketch the phase portrait for each case.

Solution:

(a) r_1, r_2 - growth rates

K_1, K_2 - carrying capacities

α, β - competition terms.

(b) Let

$$D = N_1/K_1, \quad R = N_2/K_2, \quad \tau = r_1 t$$

$$\Rightarrow \dot{D} = D(1-D) - aDR,$$

$$\dot{R} = bR(1-R) - cDR$$

where $a = \alpha K_2/r_1$, $b = r_2/r_1$, $c = \beta K_1/r_1$.

(c) The nullclines satisfy

N1 ($\dot{D}=0$):

$$D = 0$$

N2 ($\dot{D}=0$):

$$R = (1-D)/a$$

N3 ($\dot{R}=0$):

$$R = 0$$

N4 ($\dot{R}=0$):

$$D = b(1-R)/c$$

Therefore, there are fixed points at $(0,0)$, $(1,0)$, $(0,1)$ and potentially when

$$b(1-(1-D)/a) - cD = 0$$

$$\Rightarrow ba - (1-D) - caD = 0$$

$$\Rightarrow ba - 1 + (1-ca)D = 0$$

$$\Rightarrow D = \frac{1-ba}{1-ca}, \quad R = \frac{(1-D)}{a} = \frac{1}{a}\left(\frac{1-ca-1+ba}{1-ca}\right) = \frac{b-c}{1-ca}$$

Therefore, letting $d = 1/a$ this fixed point is given by

$$D = \frac{d-b}{d-c}, \quad R = d\,\frac{b-c}{d-c}.$$

Consequently, this fixed point exists in the first quadrant if

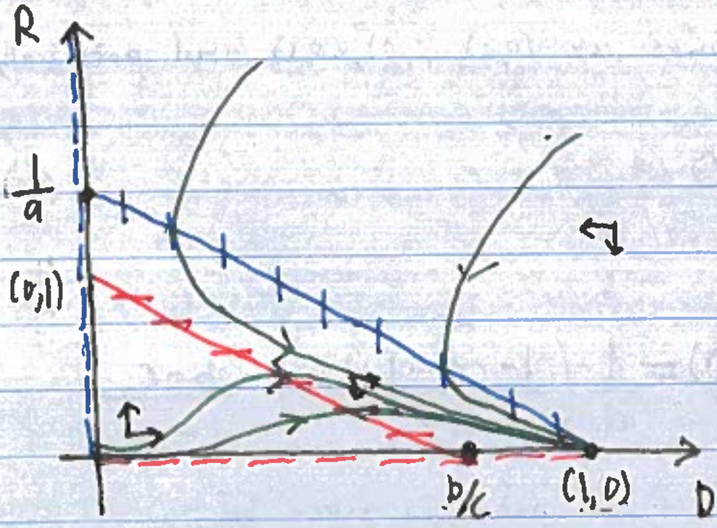
- $d > b$, $d > c$, and $b > c \Rightarrow d > b > c$
- $d < b$, $d < c$, and $b < c \Rightarrow d < b < c$

The Jacobian is given by

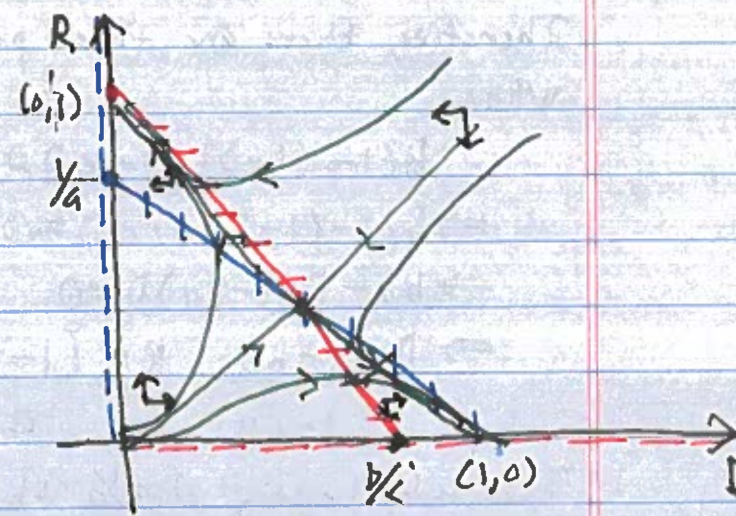
$$J(D,R) = \begin{bmatrix} 1-2D-aR & -aD \\ -cR & b-2bR-cD \end{bmatrix}$$

$$\Rightarrow J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}, \quad J(1,0) = \begin{bmatrix} -1 & -a \\ 0 & b-c \end{bmatrix}, \quad J(0,1) = \begin{bmatrix} 1-a & 0 \\ -c & -b-c \end{bmatrix}$$

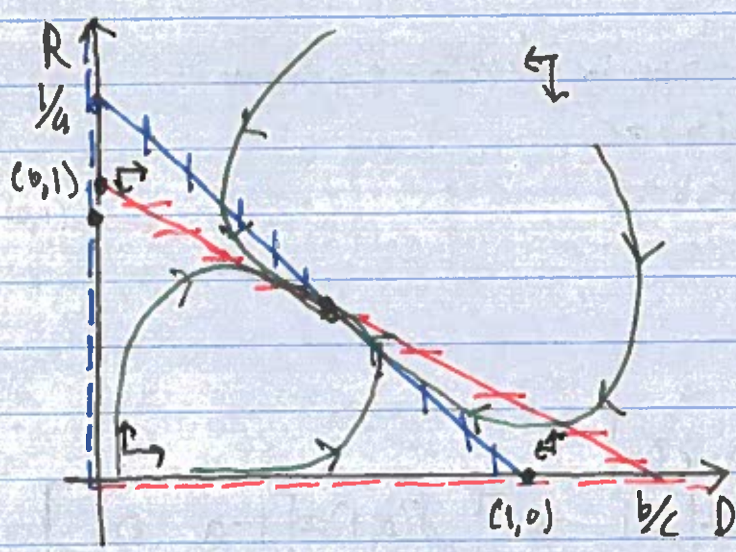
Therefore, $(0,0)$ is always unstable, $(1,0)$ is stable if $c > b$ and $(0,1)$ is stable if $a > 1$. Therefore, we have the following cases:



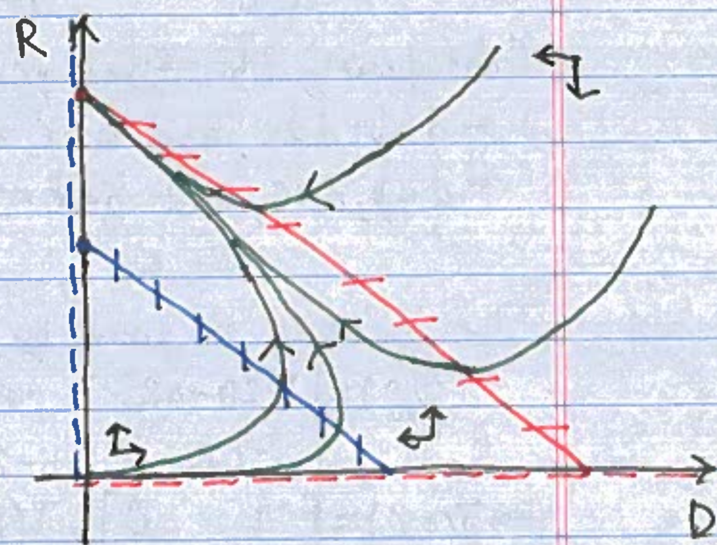
Case 1: $b/c < 1$, $1/a > 1$



Case 2: $b/c < 1$, $1/a < 1$



Case 3: $b/c > 1$, $1/a > 1$



Case 4: $b/c > 1$, $1/a < 1$

∎

#4.

Analyze the system

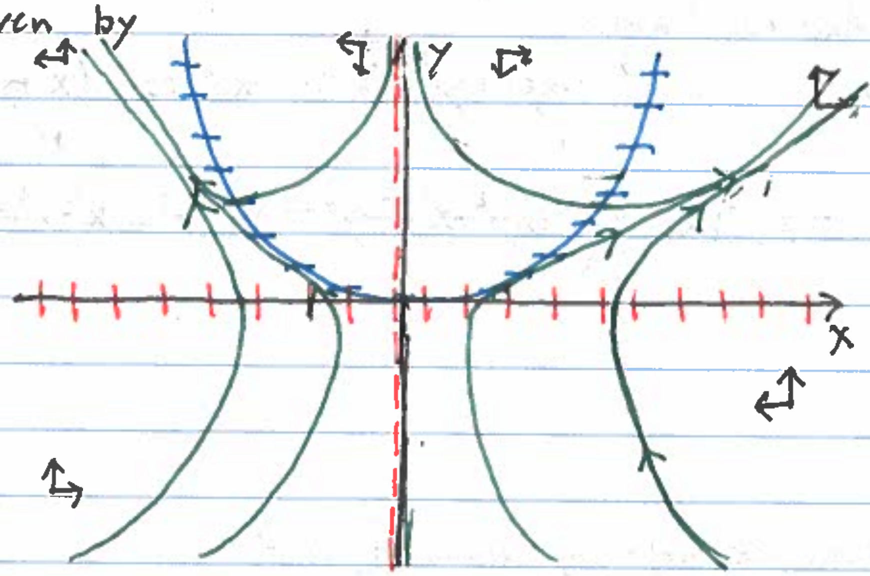
$$\dot{x} = xy$$
$$\dot{y} = x^2 - y$$

Solution:

The Jacobian at $(0,0)$ is given by:

$$J(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

Which corresponds to a line of fixed points. However it is clear that $(0,0)$ is the only fixed point. The phase portrait is given by



#5.

Consider the system $\dot{x} = -y - x^3$, $\dot{y} = x$. Prove that the original system is a spiral.

Solution:

The Jacobian at $(0,0)$ is given by

$$J(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which predicts a center. However,

$$\dot r = \frac{x\dot x + y\dot y}{r} = \frac{-xy - x^4 + xy}{r} = -r^3 < 0$$

$\cos 4\theta$

and thus trajectories go to 0.

■

#6.

Sketch a phase portrait for the system

$$\dot x = xy - x^2y + y^3$$
$$\dot y = y^2 + x^3 - xy^2.$$

Solution:

Converting to polar coordinates:

$$\dot r = \frac{x\dot x + y\dot y}{r} = \frac{x^2y - \cancel{x^3y} + \cancel{xy^3} + y^3 + \cancel{x^3y} - \cancel{xy^3}}{r} = \frac{y(x^2+y^2)}{r} = r^2\sin\theta.$$

$$\dot\theta = \frac{x\dot y - y\dot x}{r^2} = \frac{xy^2 + x^4 - x^2y^2 - xy^2 + x^2y^2 - y^4}{r^2} = \frac{x^4 - y^4}{r^2} = x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta)$$

Therefore,

$$\dot r = r^2\sin\theta$$
$$\dot\theta = r^2\cos 2\theta.$$

and thus we have the following phase portrait: