MTH 351/651
Homework #8

Due Date: November 11, 2022

1 Problems for Everyone

2 1. For the following conservative systems find all the equilibrium points and classify them, find a conserved quantity, sketch the phase portrait, find explicit formulas for any homoclinic or heteroclinic orbits.

   (a) $\ddot{x} = x^3 - x$,

   (b) $\ddot{x} = x - x^2$.

2. The relativistic equation for the orbit of a planet around the sun is

$$\frac{d^2u}{d\theta^2} + u = \alpha + \varepsilon u^2,$$

   where $u = 1/r$ and r, θ are polar coordinates of the planet in its plane of motion. The parameter α is positive and εu^2 is Einstein's correction. Here ε is a very small positive number.

   (a) Rewrite the system in the (u, v) phase plane, where $v = \frac{du}{d\theta}$.

   (b) Find all the equilibrium points of the system.

   (c) Show that one of the equilibrium is a center in the (u, v) phase plane, according to the linearization. Is it a nonlinear center?

   (d) Show that the equilibrium point found in (c) corresponds to a circular planetary orbit.

2 3. The Duffing oscillator is described by the following differential equation

$$\ddot{x} + x + \varepsilon x^3 = 0.$$

   (a) Show that this system has a nonlinear center at the origin for $\varepsilon > 0$.

   (b) If $\varepsilon < 0$, show that all trajectories near the origin are closed. What about trajectories that are far from the origin?

2 4. For each of the following systems, locate the fixed points and calculate the index.

   (a) $\dot{x} = x^2$, $\dot{y} = y$

   (b) $\dot{x} = y - x$, $\dot{y} = x^2$

   (c) $\dot{x} = y^3$, $\dot{y} = x$

   (d) $\dot{x} = xy$, $\dot{y} = x + y$

1 5. A closed orbit in the phase plane encircles S saddles, N nodes, F spirals, and C centers, all of the usual type. Show that

$$N + F + C = 1 + S.$$

1 6. A smooth vector field on the phase plane is known to have exactly three closed orbits. Two of the cycles, say C_1 and C_2, lie inside the third cycle C_3. However, C_1 does not lie inside C_2, nor vice-versa.

(a) Sketch the arrangement of the three cycles.

(b) Show that there must be at least one fixed point in the region bounded by C_1, C_2, and C_3.

7. Consider a smooth vector field $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$ on the plane, and let C be a simple closed curve that does not pass through any fixed points. As usual, let $\phi = \tan^{-1}(\dot{y}/\dot{x})$.

(a) Show that $d\phi = (f dg - g df)/(f^2 + g^2)$.

(b) Derive the following integral formula for the index

$$I_C = \frac{1}{2\pi} \int_C \frac{f dg - g df}{f^2 + g^2}.$$

8. Consider the following system of differential equations

$$\dot{x} = x - y - x(x^2 + 5y^2),$$
$$\dot{y} = x + y - y(x^2 + y^2).$$

(a) Classify the fixed point at the origin.

(b) Rewrite the system in polar coordinates.

(c) Determine the maximum radius centered at the origin such that all trajectories have a radially outward component on it.

(d) Determine the circle of minimum radius centered at the origin such that all trajectories have a radially inward component on it.

(e) Prove that the system has a limit cycle.

2 9. Consider the system

$$\dot{x} = x(1 - 4x^2 - y^2) - \frac{1}{2}y(1 + x),$$
$$\dot{y} = y(1 - 4x^2 - y^2) + 2x(1 + x).$$

(a) Show that the origin is an unstable fixed point.

(b) By considering $\dot{V}$, where $V = (1 - 4x^2 - y^2)^2$, show that all trajectories approach the ellipse $4x^2 + y^2 = 1$ as $t \to \infty$.

Homework #8

#1.

For the following conservative systems find all the equilibrium points and classify them, find a conserved quantity, sketch the phase portrait, find explicit formulae for any homoclinic or heteroclinic orbits.

(a) $\ddot{x} = x^3 - x$

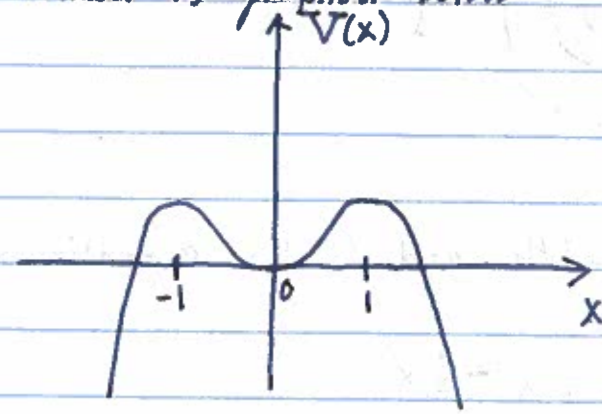
(b) $\ddot{x} = x - x^2$

Solution:

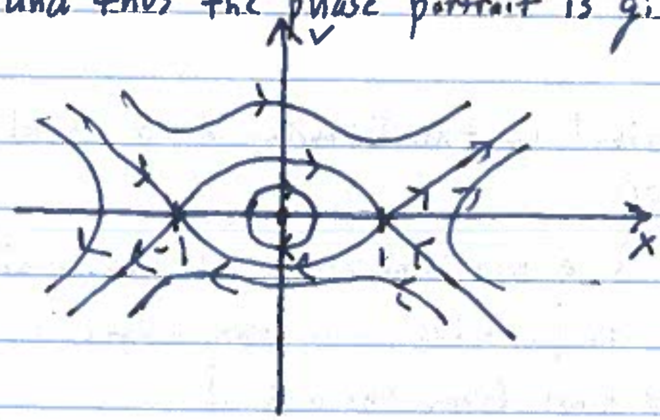
(a) The potential for this system is given by

$$V(x) = x^2/2 - x^4/4.$$

which is graphed below:



and thus the phase portrait is given by



Consequently, $(0,0)$ is a (nonlinear center), while $(\pm 1, 0)$ are saddles. The heteroclinic orbits satisfy

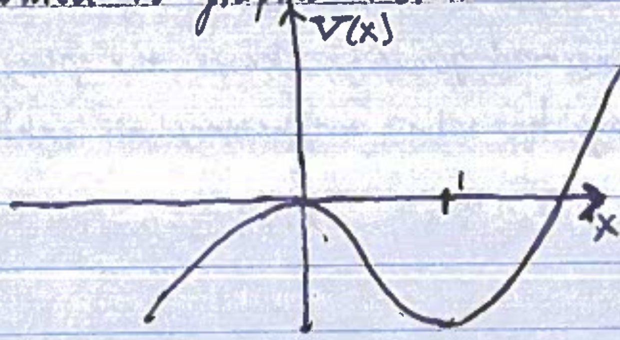
$$E(1,0) = 1/2 - 1/4 = \frac{1}{2}v^2 + x^2/2 - x^4/4$$

$$\Rightarrow 1/4 = \frac{1}{2}v^2 + x^2/2 - x^4/4.$$

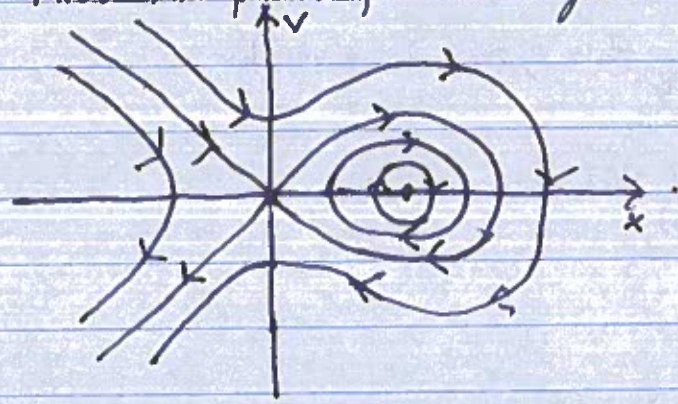
(b) The potential for this system is given by

$$V(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2$$

which is graphed below



and thus the phase portrait is given by



Consequently, $(0,0)$ is a saddle and $(1,0)$ is a nonlinear center. The homoclinic orbit satisfies

$$E(0,0) = 0 = \frac{1}{2}v^2 + \frac{1}{3}x^3 - \frac{1}{2}x^2.$$

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#3.

The Duffing oscillator is described by the following differential equation

$$\ddot{x} + x + \varepsilon x^3 = 0$$

(a) Show that this system has a nonlinear center at the origin for $\varepsilon > 0$.

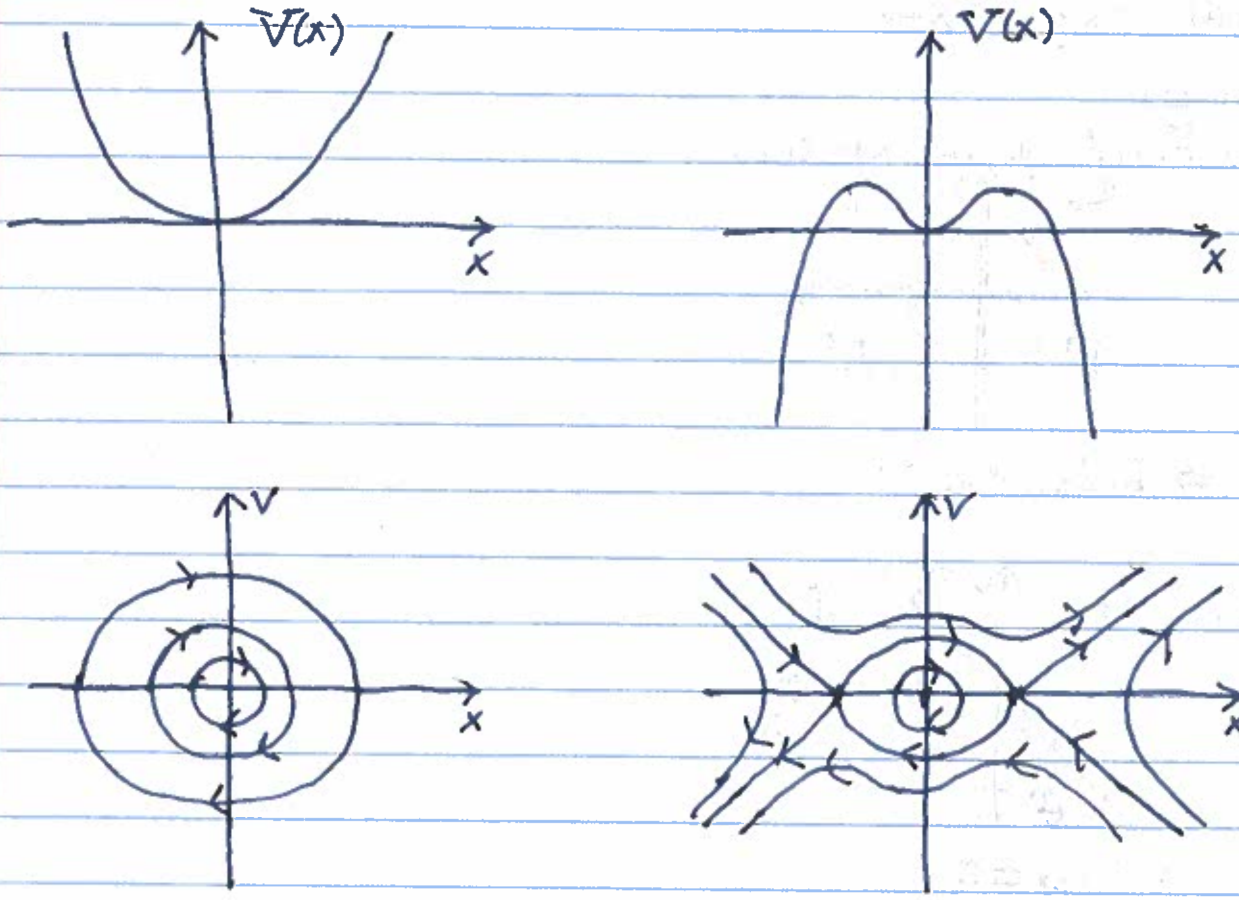
(b) If $\varepsilon < 0$, show that all trajectories near the origin are closed. What about trajectories far from the origin?

Solution:

The potential for this system is given by:

$$V(x) = \frac{\varepsilon x^4}{4} + \frac{x^2}{2}$$

The two cases are plotted below with their corresponding phase portraits:



$\varepsilon > 0$ $\quad\quad$ $\varepsilon < 0$

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#4.

For each of the following systems locate the fixed points and calculate the index.

(a) $\dot{x}=x^2, \dot{y}=y$

(b) $\dot{x}=y-x, \dot{y}=x^2$

(c) $\dot{x}=y^3, \dot{y}=x$

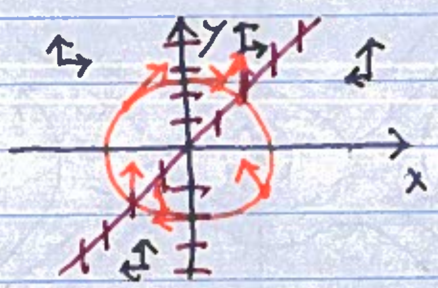
(d) $\dot{x}=xy, \dot{y}=x+y$

Solution:

(a). Fixed point at $(0,0)$

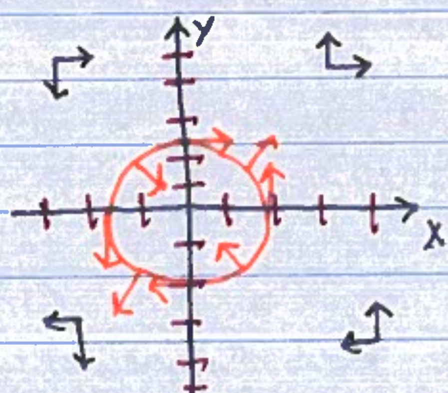
$\Rightarrow$ Index $=0$.

(b)

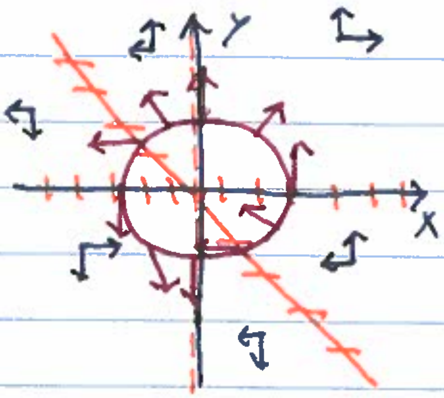


$\Rightarrow$ Index $=0$.

(c)



$\Rightarrow$ Index $=-1$.

(d) 

$\Rightarrow$ Index $= 0$

#5

A closed orbit in the phase plane encircles S saddles, N nodes F spirals, and G centers, all of the usual type. Show that

$$N + F + C = 1 + S.$$

Solution:

Since the index of a closed orbit is 1 it follows that

$$N + F + C - S = 1$$

$$\Rightarrow N + F + C = 1 + S.$$

#6.

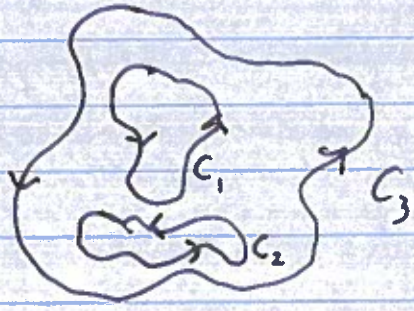
A smooth vector field on the phase plane is known to have exactly three closed orbits. Two of the cycles, say C_1 and C_2, lie inside the third cycle C_3. However, C_1 does not lie inside C_2 nor vice-versa.

(a) Sketch the arrangement of the three cycles.

(b) Show that there must be at least one fixed point in the region bounded by C_1, C_2, and C_3.

Solution:

(a)



(b). Since the index of C_3 is 1 and the index of C_1 and C_2 are also 1 there must exist another fixed point of index -1 lying outside of C_1 and C_2.

#9.

Consider the system

$$\dot{x} = x(1-4x^2-y^2) - \frac{1}{2}y(1+x),$$
$$\dot{y} = y(1-4x^2-y^2) + 2x(1+x),$$

(a) Show that the origin is an unstable fixed point.

(b) By considering $\dot{V}$, where $V = (1-4x^2-y^2)^2$, show that trajectories approach the ellipse $4x^2+y^2=1$ as $t \to \infty$.

Solution:

(a) The Jacobian at $(0,0)$ is given by

$$J(0,0) = \begin{bmatrix} 1 & -1/2 \\ 2 & 1 \end{bmatrix}$$

$$\Rightarrow \lambda_1 + \lambda_2 = 2 \quad \lambda_1\lambda_2 = 2$$
$$\Rightarrow \lambda_1 = 2 - \lambda_2$$
$$\Rightarrow (2-\lambda_2)\lambda_2 = 2$$
$$\Rightarrow \lambda^2 - 2\lambda_2 + 2 = 0$$
$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

and thus the origin is an unstable spiral.

(b) Differentiating we have that

$$\dot{V} = 2(1-4x^2-y^2)(-8x\dot{x}-2y\dot{y})$$

$$= 2(1-4x^2-y^2)(-8x^2V^{1/2}+4xy(1+x)-2y^2V^{1/2}-4xy(1+x))$$

$$= 2V(-8x^2-2y^2)$$

$$= 2V(2(1-4x^2-2y^2)-2)$$

$$= 4V(1-V^{1/2})$$

This system has a stable fixed point at $V=1$ and thus $\lim_{t\to\infty} V(t) = 1$.