MTH 351/651
Homework #8

Due Date: December 02, 2022

1 Problems for Everyone

1. Prove that the system $\dot{x} = x - y - x^3$, $\dot{y} = x + y - y^3$ has a periodic solution.
2. Show that the system $\dot{x} = -x - y + x(x^2 + 2y^2)$, $\dot{y} = x - y + y(x^2 + 2y^2)$ has at least one periodic solution.
3. Discuss the bifurcations of the system

$$\begin{aligned}\dot{r} &= r(\mu - \sin(r))\\ \dot{\theta} &= 2\mu - \sin(\theta)\end{aligned}$$

   as μ varies. Here, r and θ represent the standard polar coordinates.
4. Consider the following modified version of the predator prey system:

$$\begin{aligned}\dot{x} &= x(x(1-x) - y),\\ \dot{y} &= y(x - a),\end{aligned}$$

   where $a \geq 0$.
   (a) Sketch the nullclines in the first quadrant $x, y \geq 0$
   (b) Show that the fixed points are $(0,0)$, $(1,0)$, and $(a, a - a^2)$, and classify them.
   (c) Show that a Hopf bifurcation occurs at $a_c = 1/2$. Is it subcritical or supercritical?
   (d) Sketch all the topologically different phase portraits for $0 < a < 1$ and interpret them in practical terms.
5. Consider the following dynamical system on the torus:

$$\begin{aligned}\dot{\theta}_1 &= \omega_1 + \sin(\theta_1)\cos(\theta_2),\\ \dot{\theta}_2 &= \omega_2 + \sin(\theta_2)\cos(\theta_1),\end{aligned}$$

   where $\omega_1, \omega_2 \geq 0$.
   (a) Sketch all of the qualitatively different phase portraits that arise as ω_1, ω_2 vary.
   (b) Find the curves in ω_1, ω_2 parameter space along which bifurcations occur, and classify the various bifurcations.
   (c) Plot the stability diagram in ω_1, ω_2 parameter space.

Homework #8

#1

Prove that the system

$$\dot{x} = x - y - x^3$$
$$\dot{y} = x + y - y^3$$

has a periodic solution.

Solution:

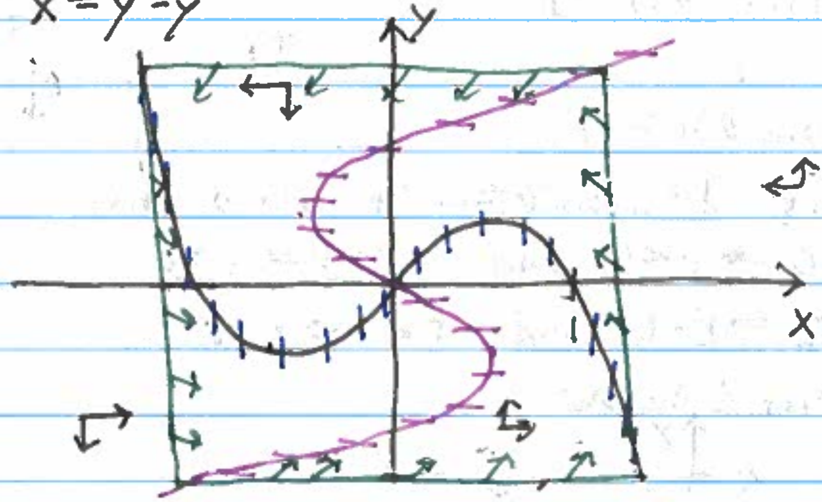
The null lines for this system are drawn below

$\dot{x} = 0$:

$$y = x - x^3$$

$\dot{y} = 0$:

$$x = y^3 - y$$



The Jacobian at the origin is given by

$$J(0,0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \lambda_{1,2} = 1 \pm i$$

and thus the origin is an unstable spiral. Therefore, considering the green square above, $(0,0)$ is an unstable fixed point contained in a trapping region. Therefore, there exists a limit cycle containing the origin.

#2.

Show that the system

$$\dot{x} = -x - y + x(x^2 + 2y^2)$$
$$\dot{y} = x - y + y(x^2 + 2y^2)$$

has at least one periodic solution.

Solution:

Converting to polar coordinates we have that

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{-x^2 - xy + x^2(x^2+2y^2) + xy - y^2 + y^2(x^2+2y^2)}{r}$$

$$\Rightarrow \dot{r} = \frac{-r^2 + r^2(r^2 + r^2\sin^2\theta)}{r}$$

$$\Rightarrow \dot{r} = \frac{r^2\left((1+\sin^2\theta)r^2 - 1\right)}{r}$$

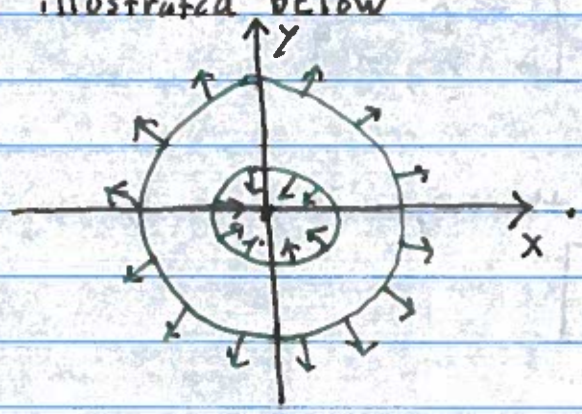
$$= r\left((1+\sin^2\theta)r^2 - 1\right)$$

Therefore, since $1 \le 1+\sin^2\theta \le 2$ it follows that

$$2r^2 - 1 < 0 \Rightarrow \dot{r} < 0 \quad \text{and} \quad r^2 - 1 > 0 \Rightarrow \dot{r} > 0$$

$$\Rightarrow r < 1/\sqrt{2} \Rightarrow \dot{r} < 0 \quad \text{and} \quad r > 1 \Rightarrow \dot{r} > 0$$

which is illustrated below



Therefore, since we have a repelling region without a fixed point there must exist an unstable limit cycle.

#3

Discuss the bifurcations of the system

$$\dot{r} = r(\nu - \sin(r))$$
$$\dot{\theta} = 2\nu - \sin\theta$$

as ν varies. Here, r and θ represent the standard polar coordinates.

Solution:

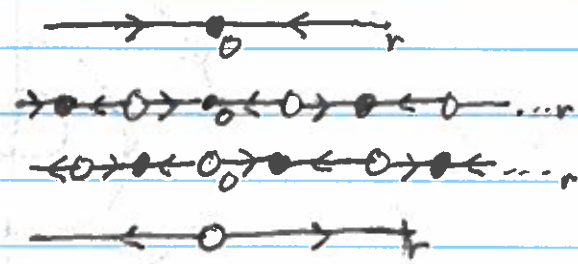
Both of these equations can be decoupled allowing us to analyze r and θ separately.

$\nu < -1$:

$-1 < \nu < 0$:

$0 < \nu < 1$:

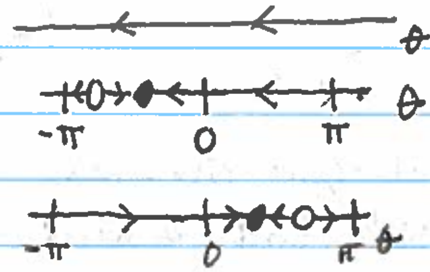
$\nu > 1$:



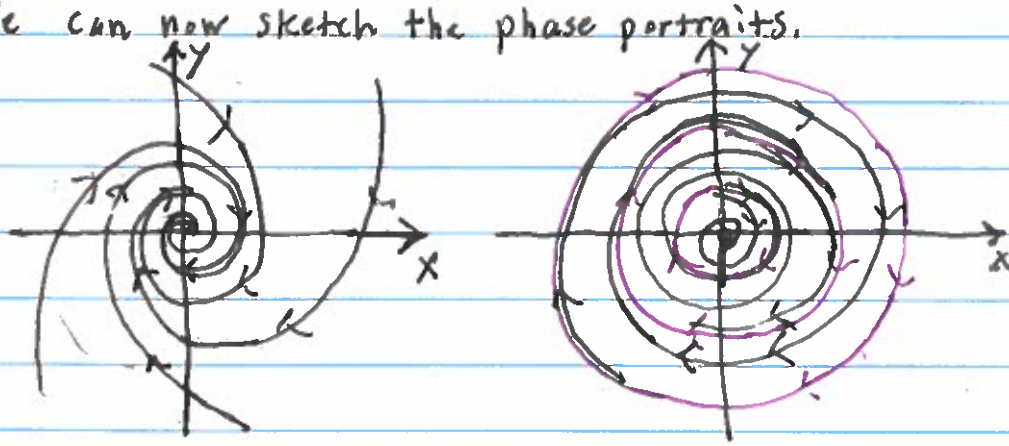
$\nu < -\frac{1}{2}$:

$-\frac{1}{2} < \nu < 0$:

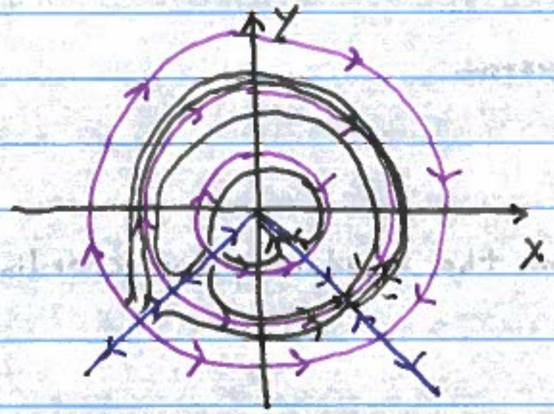
$0 < \nu < \frac{1}{2}$:



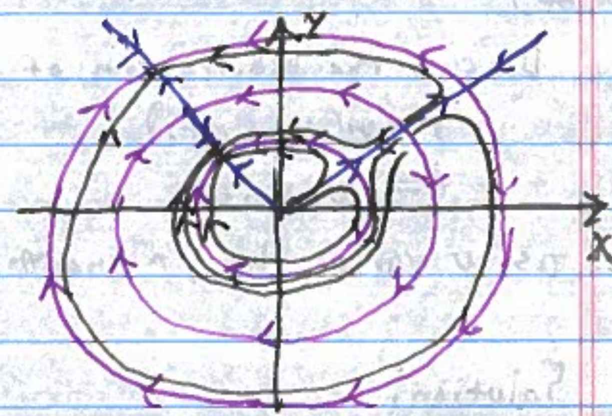
We can now sketch the phase portraits.



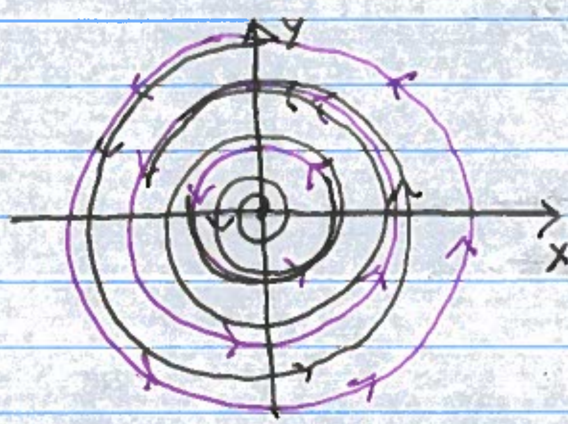
$\nu < -1$ $\qquad\qquad$ $-1 < \nu < -\frac{1}{2}$



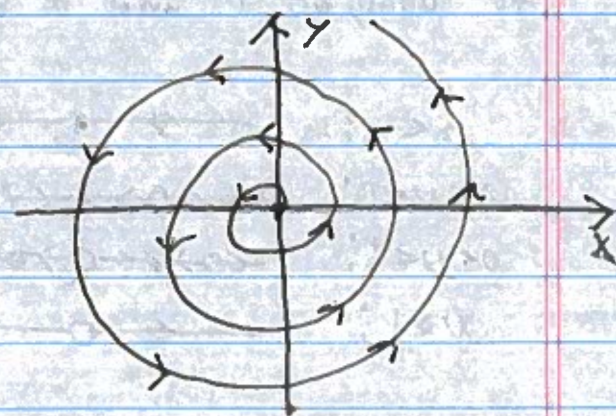
$-\frac{1}{2} < \mu < 0$



$0 < \mu < \frac{1}{2}$



$\frac{1}{2} < \mu < 1$



$1 < \mu$

In summary, there are saddle-cycle bifurcations at $\mu = \pm 1$, infinite period bifurcations at $\mu = \pm \frac{1}{2}$ and a weird bifurcation at $\mu = 0$ in which stability of the origin changes from an unstable to stable spiral without generating a limit cycle.

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#4

Consider the following modified version of the predator prey system:

$$\dot{x} = x(x(1-x)-y)$$
$$\dot{y} = y(x-a)$$

where $a \geq 0$.

(a) Sketch the nullclines in the first quadrant $x, y \geq 0$.

(b) Show that the fixed points are $(0,0)$, $(1,0)$, and $(a, a-a^2)$, and classify them.

(c) Show that a Hopf bifurcation occurs at $a_c = \frac{1}{2}$. Is it subcritical or supercritical.

(d) Sketch all the topologically differeric phase permits for $0 < a < 1$ and interpret them in practical terms.

Solution:

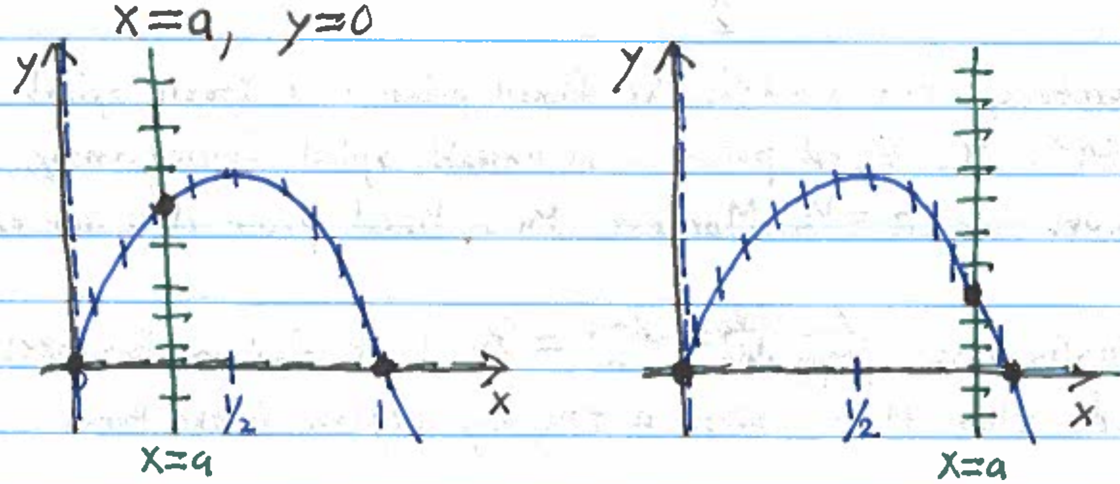
(a) The nullclines are given by

$\dot{x} = 0$:

$$y = x(1-x), \quad x = 0$$

$\dot{y} = 0$:

$$x = a, \quad y = 0$$



$a < \frac{1}{2}$ $\qquad\qquad$ $a > \frac{1}{2}$

(b) The fixed points are at $(0,0)$, $(1,0)$, $(a, a-a^2)$. The Jacobian is given by

$$J(x,y) = \begin{bmatrix} 2x-3x^2-y & -x \\ y & x-a \end{bmatrix}$$

$$\Rightarrow J(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -a \end{bmatrix} \Rightarrow \text{Indeterminate Fixed Point.}$$

$$J(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & 1-a \end{bmatrix} \Rightarrow \text{Stable node if } a>1\text{, saddle if } a<1$$

$$J(a, a-a^2) = \begin{bmatrix} 2a-3a^2-a+a^2 & -a \\ a-a^2 & 0 \end{bmatrix} = \begin{bmatrix} a-2a^2 & -a \\ a-a^2 & 0 \end{bmatrix}$$

$$\lambda_{1,2} = \frac{a-2a^2 \pm \sqrt{(a-2a^2)^2+4a^2(1-a)}}{2}$$

$$= \frac{a-2a^2 \pm \sqrt{a^2-4a^3+4a^4+4a^2-4a^3}}{2}$$

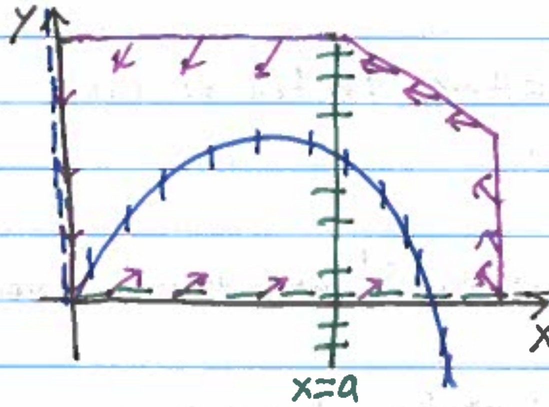
$$= \frac{a-2a^2 \pm \sqrt{5a^2+4a^4-8a^3}}{2}$$

$$= \frac{a-2a^2 \pm a\sqrt{4a^2-8a+5}}{2}$$

$$= \frac{a-2a^2 \pm a\sqrt{(2a+1)(2a-5)}}{2}$$

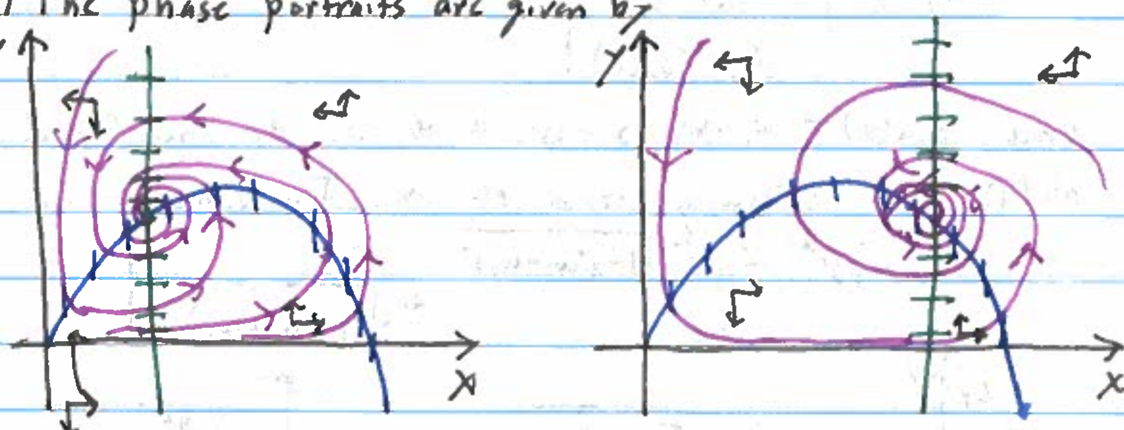
Therefore, for $0<a<1/2$ the fixed point is a stable spiral while for $1/2<a<1$ the fixed point is an unstable spiral. Consequently, a Hopf bifurcation occurs at $a_c = 1/2$. Moreover, this fixed point does not exist for $a>1$.

Finally, since $\frac{dy}{dx} = -\frac{x-a}{x} = a/x - 1$ it follows for $a<x<1$ that $a-1<\frac{dy}{dx}<0$ and thus there exists a trapping region of the form



Consequently, $a_c = 1/2$ is a supercritical Hopf bifurcation. Moreover, $a=1$ is a transcritical bifurcation.

(d) The phase portraits are given by



$a < 1/2$ $\qquad$ $a > 1/2$

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#5

Consider the following dynamical system on the torus

$$\dot{\theta}_1 = \omega_1 + \sin(\theta_1)\cos(\theta_2),$$
$$\dot{\theta}_2 = \omega_2 + \sin(\theta_2)\cos(\theta_1),$$

(a) Sketch all of the qualitatively different phase portraits that arise as ω_1, ω_2 vary.

(b) Find the curves in ω_1, ω_2 parameter space along which bifurcations occur, and classify the various bifurcations.

(c) Plot the stability diagram in ω_1, ω_2 parameter space.

Solution:

(a) Letting $x = \theta_1 - \theta_2$ and $y = \theta_1 + \theta_2$ it follows that

$$\dot{x} = \omega_1 - \omega_2 + \sin(x)$$
$$\dot{y} = \omega_1 + \omega_2 + \sin(y)$$

These are decoupled systems on S^1 and thus we have the following cases

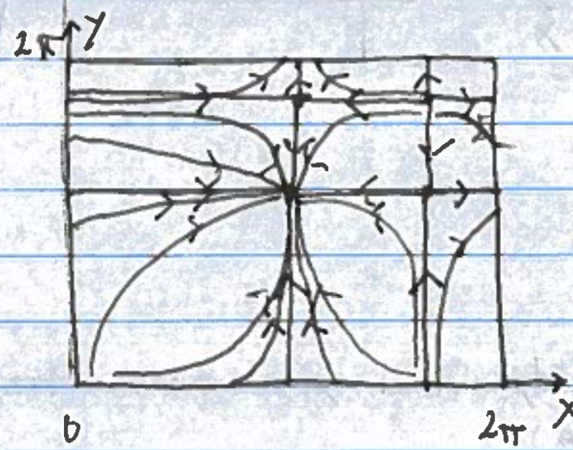
- $-1 < \omega_1 - \omega_2 < 1$, $-1 < \omega_1 + \omega_2 < 1 \Rightarrow$ four fixed points
- $-1 < \omega_1 - \omega_2 < 1$, $|\omega_1 + \omega_2| > 1 \Rightarrow$ two limit cycles
- $|\omega_1 - \omega_2| > 1$, $|\omega_1 + \omega_2| > 1 \Rightarrow$ no invariant sets.

Since

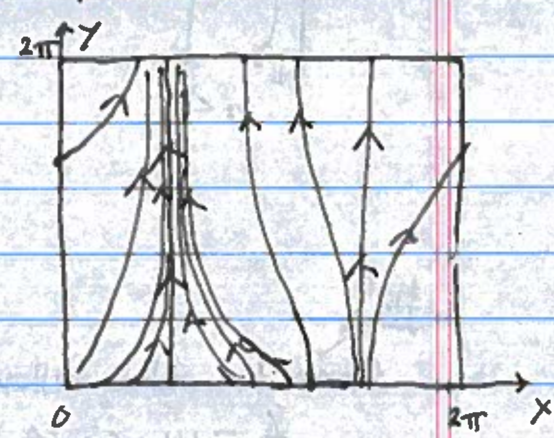
$$J(x,y) = \begin{bmatrix} \cos(x) & 0 \\ 0 & \cos(y) \end{bmatrix}$$

and $\sin(x) = \omega_2 - \omega_1$, $\sin(y) = -\omega_1 - \omega_2$ at the fixed points we obtain the following form of the Jacobian at all fixed points (x^*, y^*):

$$J(x^*, y^*) = \begin{bmatrix} \pm\sqrt{1-(\omega_2-\omega_1)^2} & 0 \\ 0 & \pm\sqrt{1-(\omega_1+\omega_2)^2} \end{bmatrix}$$



$|\omega_1 - \omega_2| < 1$, $|\omega_1 + \omega_2| < 1$
$\Rightarrow \theta_1, \theta_2$ go to a fixed point



$|\omega_1 - \omega_2| < 1$, $|\omega_1 + \omega_2| > 1$
$\Rightarrow \theta_1, \theta_2$ go to a limit cycle

For $|\omega_1 - \omega_2| > 1$ and $|\omega_1 + \omega_2| > 1$ we get quasiperiodicity as the torus is foliated.

In parameter space we have the following diagram