MST 352/652
Homework #1

Due Date: January 22, 2019

1 Problems for everyone

1. pg. 8, #1.5-8, #1.10, #1.11
2. pg. 13, #1.19, #1.20, #1.22, #1.24-1.26
3. For what values of a and b is the function $u(x,t) = e^{at}\sin(bx)$ a solution to the heat equation:

$$u_t = ku_{xx}.$$

2 Graduate Problems

Undergraduate students can complete these exercises for extra credit The goal of these exercises is to introduce the notion of **dispersion relationships** and **linear superposition**. Namely, you will learn how solutions to linear partial differential equations can be built up out of a **linear superposition of plane wave solutions.**

1. (**Dispersion Relationships** Linear, homogeneous partial differential equations with constant coefficients admit complex solutions of the form

$$u(x,t) = Ae^{i(kx-\omega t)}$$

which are called **plane waves**. The real and imaginary parts of this complex function give real solutions. The constant A is the **amplitude**, k is the **wave number**, and ω is the **temporal (angular) frequency**. When the plane wave form is substituted into the partial differential equation there results a dispersion relationship of the form

$$\omega = \omega(k)$$

which states how the frequency depends upon the wave number in order for the plane wave to be a solution. For the following partial differential equations find the dispersion relationship and determine the resulting plane wave:

(a) $u_t = u_{xx}$ (heat equation)

(b) $u_{tt} = u_{xx}$ (wave equation)

(c) $u_t = iu_{xx}$ (Schrödinger's equation)

2. **(Superposition of Solutions)** For the dispersion relationships determined above for the heat and Schrödinger's equation, show that

$$u(x,t) = \int_{-\infty}^{\infty} f(k) e^{i(kx - \omega(k)t)}\, dk$$

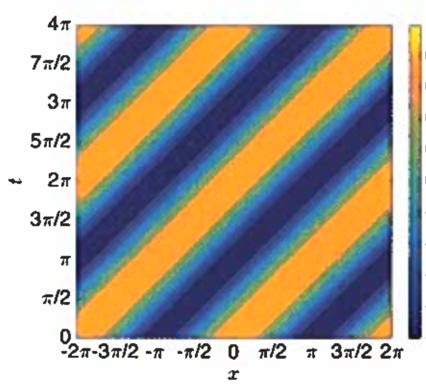
is also a solution to the partial differential equation for any function $f(k)$ that is bounded and continuous on $[0, \infty)$. This exercise illustrates that more complicated solutions to partial differential equations can be built up from plane waves.

3. **Backwards** Figures (a-c) below are contour plots for the *real component* of plane wave solutions $u(x,t) = e^{i(kx - \omega(k)t)}$ to an unknown linear partial differential equation.

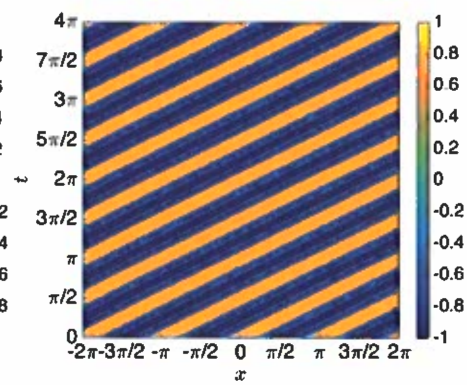
   (a) For each figure determine the value of k and ω. **Hint:** Think about how k and ω are related to spatial wavelength and temporal period.

   (b) Make a conjecture of the dispersion relationship $\omega(k)$.

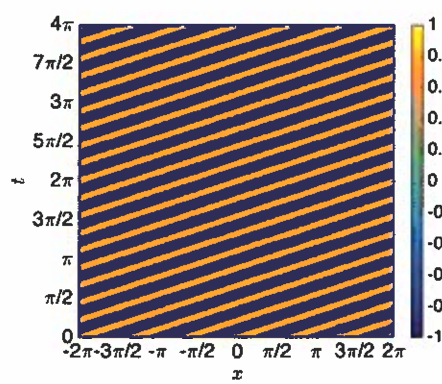
   (c) From this dispersion relationship determine the partial differential equation these plane wave solutions satisfy.



(a)



(b)



(c)

Historical note: This particular partial differential equation was derived by physicists in this exact manner. Experiments could measure a relationship between spatial wavelength and temporal frequency which when combined gave a dispersion relationship and equivalently a partial differential equation!

Homework #1

pg. 8, #1.5

Show that the following functions $u(x,y)$ define classical solutions to Laplace's equation $u_{xx}+u_{yy}=0$:

a.) $e^x\cos(y)$
b.) $1+x^2-y^2$
c.) x^3-3xy^2
d.) $\ln(x^2+y^2)$
e.) $\tan^{-1}(y/x)$
f.) $\dfrac{x}{x^2+y^2}$

Solution:

a.) Let $u(x,y)=e^x\cos(y)$. It follows that

$$U_x=e^x\cos(y)$$
$$U_{xx}=e^x\cos(y)$$
$$U_y=-e^x\sin(y)$$
$$U_{yy}=-e^x\cos(y)$$

Therefore, on all of $\mathbb{R}^2$ $u(x,y)$ satisfies Laplace's equation.

d) Let $u(x,y)=\ln(x^2+y^2)$. It follows that

$$U_x=\frac{2x}{x^2+y^2}$$
$$U_{xx}=\frac{(x^2+y^2)2-4x^2}{(x^2+y^2)^2}=\frac{-2x^2+2y^2}{(x^2+y^2)^2}$$
$$U_y=\frac{2y}{x^2+y^2}$$
$$U_{yy}=\frac{2x^2-2y^2}{(x^2+y^2)^2}$$

Therefore, on $\mathbb{R}^2/\{(0,0)\}$ $u(x,y)$ satisfies Laplace's equation.

e) Let $u(x,y)=\tan^{-1}(y/x)$. It follows that

$$u_x = \frac{1}{1+y^2/x^2}\cdot\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$u_{xx} = \frac{2xy}{(x^2+y^2)^2}$$

$$u_y = \frac{1}{1+y^2/x^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$u_{yy} = -\frac{2xy}{(x^2+y^2)^2}$$

However, $\tan^{-1}\left(\frac{y}{x}\right)$ is not defined when $y/x = \frac{\pi}{2}+n\pi$, where $n\in\mathbb{Z}$. Consequently, $u(x,y)=\tan^{-1}(y/x)$ satisfies Laplace's equation on $\mathbb{R}^2/\{(x,y): y=\pi\left(\frac{1+n}{2}\right)x, \text{ where } n\in\mathbb{Z}\}$.

f) Let $u(x,y)=\frac{x}{x^2+y^2}$. It follows that

$$u_x = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = -\frac{2xy}{(x^2+y^2)^2}$$

$$u_{xx} = \frac{-2x(x^2+y^2)^2-(y^2-x^2)2(x^2+y^2)2x}{(x^2+y^2)^4}$$

$$= \frac{-2x(x^2+y^2)-4x(y^2-x^2)}{(x^2+y^2)^3}$$

$$= \frac{2x^3-6xy^2}{(x^2+y^2)^3}$$

$$u_{yy} = \frac{(x^2+y^2)^2(-2x)+2xy\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(-2x)+8xy^2}{(x^2+y^2)^3}$$

$$= \frac{-2x^3+6xy^2}{(x^2+y^2)^3}$$

Therefore, on $\mathbb{R}^2/\{0,0\}$ $u(x,y)$ satisfies Laplace's equation. ∎

pg. 8, #1.6

Find all solutions $u = f(r)$ of the two dimensional Laplace's equation.

Solution:

Let $u(x,y) = f(r)$. First, in polar coordinates

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial r}\frac{\partial}{\partial \theta} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial r}\frac{\partial}{\partial \theta} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

Therefore,

$$\frac{\partial^2}{\partial x^2} f(r) = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right) f(r)$$

$$= \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\cos\theta \frac{df}{dr}$$

$$= \cos^2\theta \frac{d^2f}{dr^2} + \frac{\sin^2\theta}{r}\frac{df}{dr}$$

$$\frac{\partial^2}{\partial y^2} f(r) = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right) f(r)$$

$$= \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\sin\theta \frac{df}{dr}$$

$$= \sin^2\theta \frac{d^2f}{dr^2} + \frac{\cos^2\theta}{r}\frac{df}{dr}.$$

Therefore,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} = 0.$$

Let $g(r) = \frac{df}{dr}$. Therefore,

$$\frac{dg}{dr} = -\frac{1}{r} g$$

$$\Rightarrow \int \frac{1}{g} dg = -\int \frac{1}{r} dr$$

$$\Rightarrow \ln(g) = -\ln(r) + c$$

$$\Rightarrow g(r) = c r^{-1}$$

Finally,

$$f(r) = \int g(r)\,dr = c\ln(r) + d.$$

pg. 8, #1.7

Find all real solutions to Laplace's equation $u_{xx}+u_{yy}=0$ of the form $u(x,y)=\ln(p(x,y))$, where $p(x,y)$ is a quadratic polynomial.

Solution:

Let $p(x,y)=a+bx+cy+dx^2+exy+fy^2$. It follows that

$$\frac{\partial}{\partial x}\ln(p(x,y))=\frac{1}{p(x,y)}\frac{\partial}{\partial x}p(x,y)=\frac{1}{p(x,y)}(b+2dx+cy)$$

$$\frac{\partial^2}{\partial x^2}\ln(p(x,y))=\frac{p(x,y)(2d)-(b+2dx+ey)^2}{p(x,y)^2}$$

$$=\frac{2ad+2bdx+2cdy+2d^2x^2+2dexy+2dfy^2-b^2-4bdx-2bey-4d^2x^2-4dexy-e^2y^2}{p(x,y)^2}$$

$$=\frac{2ad-b^2-2bdx+2(cd-be)y-2d^2x^2-2dexy+(2df-e^2)y^2}{p(x,y)^2}$$

$$\frac{\partial}{\partial y}\ln(p(x,y))=\frac{1}{p(x,y)}\frac{\partial}{\partial y}p(x,y)=\frac{1}{p(x,y)}(c+ex+2fy)$$

$$\frac{\partial^2}{\partial y^2}\ln(p(x,y))=\frac{p(x,y)(2f)-(c+ex+2fy)^2}{p(x,y)^2}$$

$$=\frac{2af+2bfx+2cfy+2dfx^2+2efxy+2f^2y^2-c^2-2cex-4cfy-e^2x^2-4efxy-4f^2y^2}{p(x,y)^2}$$

$$=\frac{2af-c^2+2(bf-ce)x-2cfy+(2df-e^2)x^2-2efxy-2f^2y^2}{p(x,y)^2}$$

Gathering coefficients we obtain the following system

$$
\begin{aligned}
&2ad-b^2+2af-c^2=0\\
&-2bd+2bf-2ce=0\\
&2cd-2be-2cf=0\\
&-2d^2+2df-e^2=0\\
&-2de-2ef=0 \quad (*)\\
&2df-e^2-2f^2=0
\end{aligned}
$$

From $(*)$ it follows that either $e=0$ or $d=-f$.

Case 1 ($e=0$):

$$
\begin{aligned}
&2ad-b^2+2af-c^2=0\\
&-2bd+2bf=0\\
&2cd-2cf=0\\
&-2d^2+2df=0\\
&2df-2f^2=0
\end{aligned}
$$

$$
\Rightarrow
\begin{aligned}
&2ad-b^2+2af-c^2=0\\
&2b(f-d)=0\\
&2c(d-f)=0\\
&2d(f-d)=0\\
&2f(d-f)=0
\end{aligned}
$$

Case 1.a. ($d=f$)

$$2ad-b^2+2ad-c^2=0$$

So if $b^2+c^2=4ad$, $e=0$ and $d=f$. Consequently,

$$p(x,y)=a+bx+cy+d(x^2+y^2)$$

with $b^2+c^2=4ad$.

Case 1.b ($d\neq f$)

$b=0$, $c=0$, $d=0$, $f=0$.

Consequently,

$$p(x,y)=a.$$

Case 2 ($d=-f$):

$$2ad-b^2-2ad-c^2=0$$
$$-2bd-2bd-2ce=0$$
$$2cd-2be+2cd=0$$
$$-2d^2-2d^2-e^2=0$$
$$-2d^2-e^2-2d^2=0$$

$$\Rightarrow -b^2-c^2=0 \qquad (**)$$
$$-4bd-2ce=0$$
$$4cd-2be=0$$
$$-4d^2-e^2=0$$
$$-2d^2-e^2-2d^2=0$$

$(**)$ implies $b=c=0$. Therefore,
$$-4d^2-e^2=0$$
which implies $d=e=0$ and thus $f=0$. Consequently, the only solution in this case is
$$p(x,y)=a.$$

In summary, from all of the cases we find
$$p(x,y)=a+bx+cy+d(x^2+y^2)$$
with $b^2+c^2=4ad$.

$\blacksquare$

#1.11

Find all polynomial solutions $p(t,x)$ of the wave equation $u_{tt}=u_{xx}$ with

a.) $\deg p \le 2$

b.) $\deg p = 3$.

Solution:

Let $p(t,x)=a+bt+cx+dt^2+etx+fx^2+gt^3+ht^2x+ktx^2+lx^3$

$$\Rightarrow p_{tt}=2d+6gt+2hx$$
$$p_{xx}=2f+2kt+6lx.$$

Matching coefficients it follows that a,b,c are free variables while

$$d=f,\ k=3g,\ h=3l.$$

$\blacksquare$

#1.20

The displacement $u(t,x)$ of a forced violin string is modeled by the PDE $u_{tt}=4u_{xx}+F(t,x)$. When the string is subjected to the forcing

$$F_1(t,x)=\cos(x)$$

the solution is $u(t,x)=\cos(x-2t)+\frac{1}{4}\cos(x)$ while when

$$F_2(t,x)=\sin(x)$$

the solution is $u(t,x)=\sin(x-2t)+\frac{1}{4}\sin(x)$. Find a solution when the forcing is

a.) $F_3(t,x)=\cos(x)-5\sin(x)$

b.) $F_4(t,x)=\sin(x-3)$

Solution:

a.) $F_3(t,x)=F_1(t,x)-5F_2(t,x)$ and hence

$$u(t,x)=\cos(x-2t)+\tfrac{1}{4}\cos(x)-5\sin(x-2t)-\tfrac{5}{4}\sin(x).$$

Solves $u_{tt}=4u_{xx}+F_3(t,x)+F_4(t,x)$.

b.) $F_4(t,x)=F_2(t,x-3)$. Consequently,

$$u(t,x)=\sin(x-3-2t)+\tfrac{1}{4}\sin(x-3)$$

Solves $u_{tt}=4u_{xx}+F_2(t,x-3)$.

$\blacksquare$

Graduate Problems

#1.

Find the dispersion relationship for the following PDEs.

a.) $u_t = u_{xx}$

b.) $u_{tt} = u_{xx}$

c.) $u_t = i u_{xx}$

Solution:

Substituting in $Ae^{i(kx-wt)}$ into the PDEs we obtain:

a.) $-iw = -k^2$

$\Rightarrow w = -ik^2$ (Heat Equation)

b.) $-w^2 = -k^2$

$\Rightarrow w = \pm k$ (Wave Equation)

c.) $-iw = -ik^2$

$\Rightarrow w = k^2$ (Schrödinger's Equation)

#2.

For the dispersion relationships for the heat and Schrödinger's equation, show that

$$u(t,x) = \int_{-\infty}^{\infty} f(k) e^{i(kx - w(k)t)} dk$$

is also a solution for any f that is bounded and continuous on $(-\infty, \infty)$.

Solution:

a.) $u_t - u_{xx} = \int_{-\infty}^{\infty} f(k)(-iw(k) + k^2) e^{i(kx-w(k)t)} dk = 0.$

b.) $u_t - iu_{xx} = \int_{-\infty}^{\infty} f(k)(-iw(k) + ik^2) e^{i(kx-w(k)t)} dk = 0.$

#3.

Based on the figures, determine a linear PDE satisfied by the plane waves.

Solution:

In figures a-c, the values of k and w are,

(i) $k = 1, 2, 3$

(ii) $w = 1, 4, 9$

Therefore, the dispersion relationship is $w = k^2$ which corresponds to Schrödinger's equation.