MTH 351/651
Homework #5

Due Date: October 07, 2022

1 Problems for Everyone

1. Consider the arms race described by

$$\dot{x} = y,$$
$$\dot{y} = -x + y.$$

   (a) How would you describe the interaction between these two nations in practical terms?

   (b) Classify the fixed point at the origin. What does this imply about the arms race?

   (c) Sketch $x(t)$ and $y(t)$ as functions of t, assuming $x(0) = 1$, $y(0) = 0$.

2. In each of the following, predict the course of the following arms races depending on the relative sizes of a and b (assume $a, b > 0$). Explain in practical terms how the countries interact, and try to think of examples of countries that might interact in this manner.

   (a) $\dot{x} = ay$ and $\dot{y} = bx$

   (b) $\dot{x} = ax + by$ and $\dot{y} = -bx - ay$

   (c) $\dot{x} = ax + by$ and $\dot{y} = bx + ay$

   (d) $\dot{x} = 0$ and $\dot{y} = ax + by$

3. Here are the official definitions of the various types of stability. Consider a fixed point $\mathbf{x}^*$ of a system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{F} : \mathbb{R}^n \mapsto \mathbb{R}^n$.

   - We say $\mathbf{x}^*$ is **attracting** if there exists a $\delta > 0$ such that if $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$ then $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*$.
   - We say $\mathbf{x}^*$ is **Liapunov stable** if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $t \geq 0$ if $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$ then $\|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon$.

   For each of the following systems, determine whether the origin is attracting, Liapunov stable, asymptotically stable, or none of the above.

   (a) $\dot{x} = y$ and $\dot{y} = -4x$

   (b) $\dot{x} = 0$ and $\dot{y} = -y$

   (c) $\dot{x} = -x$ and $\dot{y} = -5y$

   (d) $\dot{x} = x$ and $\dot{y} = y$

4. For a 2×2 matrix A prove that the eigenvalue of A satisfy

$$\lambda_{1,2} = \frac{\mathrm{Tr}(A)}{2} \frac{\pm 1}{2} \sqrt{(\mathrm{Tr}(A))^2 - 4\det(A)}.$$