MTH 351/651
Homework #8

Due Date: December 02, 2022

1 Problems for Everyone

1. Prove that the system $\dot{x} = x - y - x^3$, $\dot{y} = x + y - y^3$ has a periodic solution.
2. Show that the system $\dot{x} = -x - y + x(x^2 + 2y^2)$, $\dot{y} = x - y + y(x^2 + 2y^2)$ has at least one periodic solution.
3. Discuss the bifurcations of the system

$$\begin{aligned}\dot{r} &= r(\mu - \sin(r))\\ \dot{\theta} &= 2\mu - \sin(\theta)\end{aligned}$$

   as μ varies. Here, r and θ represent the standard polar coordinates.

4. Consider the following modified version of the predator prey system:

$$\begin{aligned}\dot{x} &= x(x(1 - x) - y),\\ \dot{y} &= y(x - a),\end{aligned}$$

   where $a \geq 0$.

   (a) Sketch the nullclines in the first quadrant $x, y \geq 0$
   (b) Show that the fixed points are $(0, 0)$, $(1, 0)$, and $(a, a - a^2)$, and classify them.
   (c) Show that a Hopf bifurcation occurs at $a_c = 1/2$. Is it subcritical or supercritical?
   (d) Sketch all the topologically different phase portraits for $0 < a < 1$ and interpret them in practical terms.

5. Consider the following dynamical system on the torus:

$$\begin{aligned}\dot{\theta}_1 &= \omega_1 + \sin(\theta_1)\cos(\theta_2),\\ \dot{\theta}_2 &= \omega_2 + \sin(\theta_2)\cos(\theta_1),\end{aligned}$$

   where $\omega_1, \omega_2 \geq 0$.

   (a) Sketch all of the qualitatively different phase portraits that arise as ω_1, ω_2 vary.
   (b) Find the curves in ω_1, ω_2 parameter space along which bifurcations occur, and classify the various bifurcations.
   (c) Plot the stability diagram in ω_1, ω_2 parameter space.