Lecture 10: Phase Plane

Big Picture:

Differential equations with two variables

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

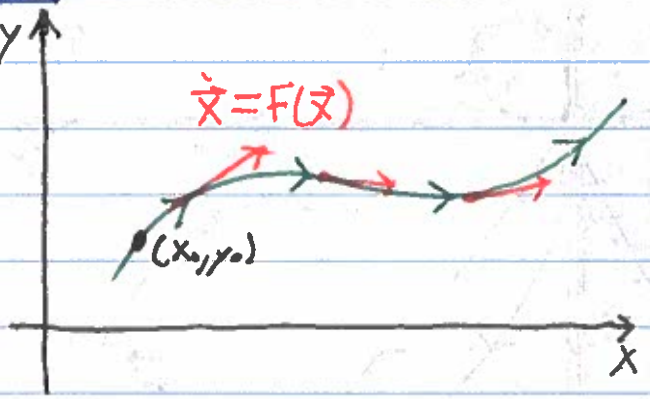
where $(x,y) \in \mathbb{R}^2$ and $f, g : \mathbb{R}^2 \to \mathbb{R}$ are smooth.

We can also write the system in the form

$$\dot{\vec{x}} = F(\vec{x})$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$

Idea: Think of solution curves as flows.



We can think of $F(\vec{x})$ as assigning a vector to each point, a solution curve is tangent to $F(\vec{x})$ at each point.

Solutions of Interest:

1. **Fixed points:** Each $\vec{x}_0$ satisfying $F(\vec{x}_0) = 0$
2. **Periodic Orbits:** $\vec{x}(t)$ is periodic if there exists $T > 0$ such that $\vec{x}(t+T) = \vec{x}(t)$ for all t and $\vec{x}(t)$ is not a fixed point.

Example:

$\dot{x} = x(x-y)$

$\dot{y} = y(2x-y)$

Nullclines are curves along which $\dot{x}=0$ or $\dot{y}=0$.

N1: $x=0$ ($\dot{x}=0$)

N2: $y=x$ ($\dot{x}=0$)

N3: $y=0$ ($\dot{y}=0$)

N4: $y=2x$ ($\dot{y}=0$)