Lecture 11: Linear Systems

Generic System:

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases} \Rightarrow \dot{\vec{x}} = A\vec{x}, \text{ with } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

1. If $\vec{x}_1, \vec{x}_2$ satisfy $\dot{\vec{x}} = Ax$ then so does $c_1\vec{x}_1 + c_2\vec{x}_2$.

proof:

$$A(c_1\vec{x}_1 + c_2\vec{x}_2) = A(c_1\vec{x}_1) + A(c_2\vec{x}_2)$$
$$= c_1 A\vec{x}_1 + c_2 A\vec{x}_2$$
$$= c_1\dot{\vec{x}}_1 + c_2\dot{\vec{x}}_2$$
$$= \frac{d}{dt}(c_1\vec{x}_1 + c_2\vec{x}_2) \quad \blacksquare$$

*To solve this system it suffices to find two linearly independent solutions to generate all solutions as linear combinations.

2. If λ is an eigenvalue with corresponding eigenvector $\vec{v}$ then

$$\vec{x} = e^{\lambda t}\vec{v}$$

is a solution.

proof:

Let $\vec{x}(t) = e^{\lambda t}\vec{v}$. Therefore,

$$\dot{\vec{x}}(t) = \lambda e^{\lambda t}\vec{v} = e^{\lambda t}(\lambda\vec{v}) = e^{\lambda t}A\vec{v} = A(e^{\lambda t}\vec{v}) = A\vec{x}.$$

3. If $\vec{v}_1, \vec{v}_2$ are linearly independent eigenvectors, then all solutions can be written in the form

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

$$\Rightarrow \vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2 \quad (\text{Solve for } c_1, c_2).$$

4. The eigendirections tell us where the flow is invariant.

Example:

1. $\dot{\vec{x}} = \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix} \vec{x} = A\vec{x}$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -2$. One eigenvector is $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. To find $\vec{v}_2$ we know

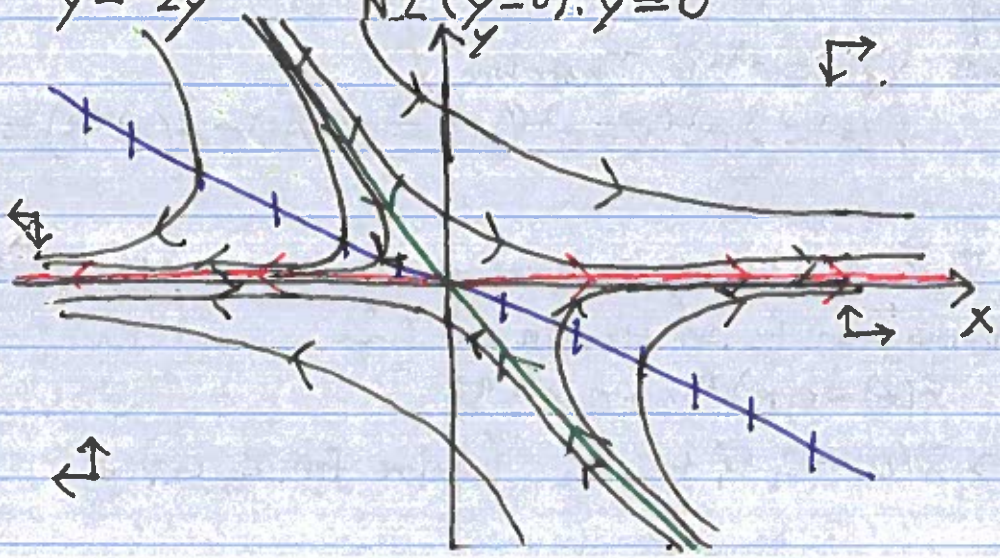
$$A\vec{v}_2 = -2\vec{v}_2$$
$$\Rightarrow (A + 2I)\vec{v}_2 = 0$$
$$\begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \vec{v}_2 = 0$$
$$\Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore,

$$\vec{X}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\Rightarrow \left.\begin{aligned} x(t) &= c_1 e^{t} + c_2 e^{-2t} \\ y(t) &= -c_2 e^{-2t} \end{aligned}\right\} \text{explicit solutions.}$$

Phase portrait:

$$\begin{aligned} \dot{x} &= x + 3y \quad \Rightarrow \text{N1 } (\dot{x} = 0): y = -x/3 \\ \dot{y} &= -2y \qquad\quad \text{N2 } (\dot{y} = 0): y = 0 \end{aligned}$$



This fixed point is classified as a saddle.

2. $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x} = A\vec{x}$

Eigenvalues satisfy:

$$\lambda_1 \lambda_2 = 1$$

$$\lambda_1 + \lambda_2 = 0$$

$$\Rightarrow \lambda_1 = i,\ \lambda_2 = -i.$$

Eigenvectors:

$$A - \lambda_1 I = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}_{+iR1} \rightarrow \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\vec{x}(t) = c_1 e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\Rightarrow \underline{x(t) = c_1 \cos(t) + i c_1 \sin(t) + c_2 \cos(t) - i c_2 \sin(t)}$$

$$\underline{y(t) = c_1 i \cos(t) - c_1 \sin(t) - c_2 i \cos(t) - c_2 \sin(t)}$$

$$\Rightarrow x(t) = (c_1 + c_2)\cos(t) + i(c_1 - c_2)\sin(t)$$

$$y(t) = (-c_2 - c_1)\sin(t) + i(c_1 - c_2)\cos(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = (c_1 + c_2) \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + i(c_1 - c_2) \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

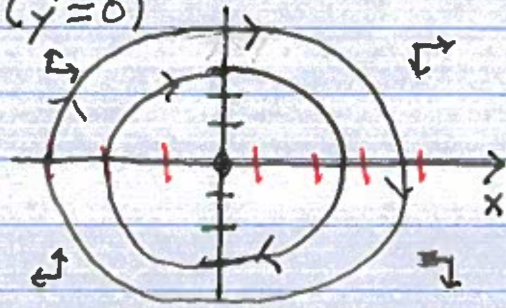
$$= \tilde{c}_1 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + \tilde{c}_2 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

$$\Rightarrow \vec{x}(t) = \tilde{c}_1 \, \mathrm{Re}\left(e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} \right) + \tilde{c}_2 \, \mathrm{Im}\left(e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} \right)$$

Phase Portrait:

N1: $y=0$ ($\dot{x}=0$)

N2: $x=0$ ($\dot{y}=0$)

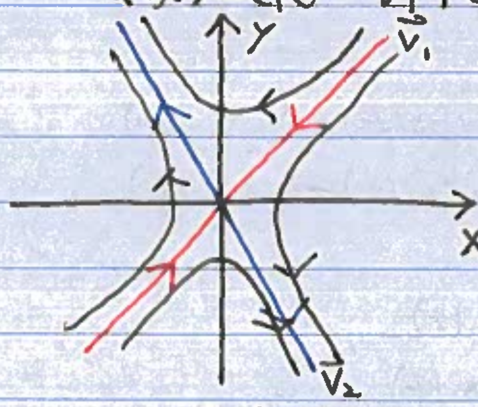


The fixed point (0,0) is called a center.

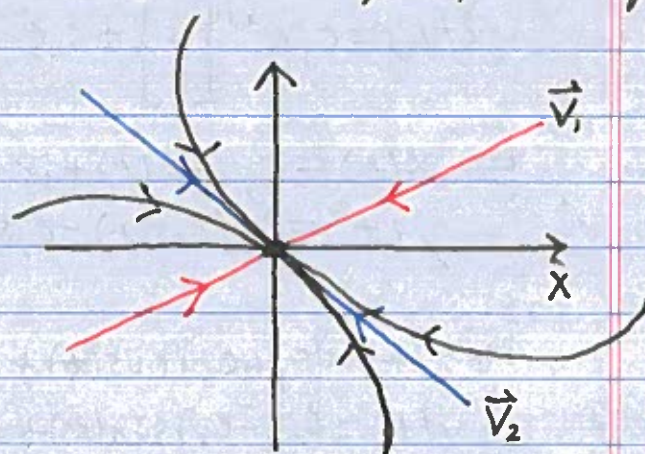
Eigenvalue Analysis

1. $\dot{\vec{x}}=A\vec{x}$, A has two real eigenvalues, two linearly dependent eigenvectors.

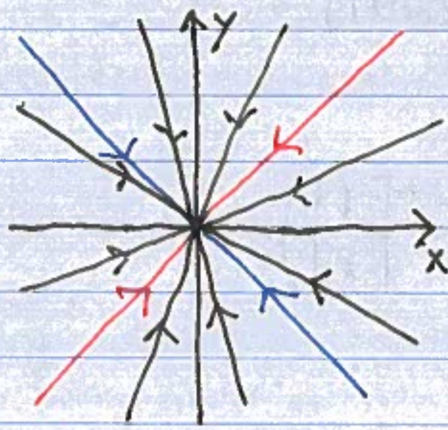
$\vec{x}(t)=c_1e^{\lambda_1 t}\vec{v}_1+c_2e^{\lambda_2 t}\vec{v}_2$



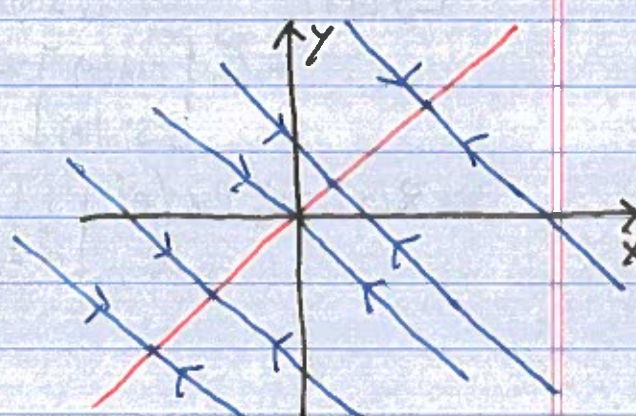
$\lambda_1<0<\lambda_2$ (Saddle Node)



$\lambda_1<\lambda_2<0$ (Stable Node)



$\lambda_1=\lambda_2<0$ (Star Node)



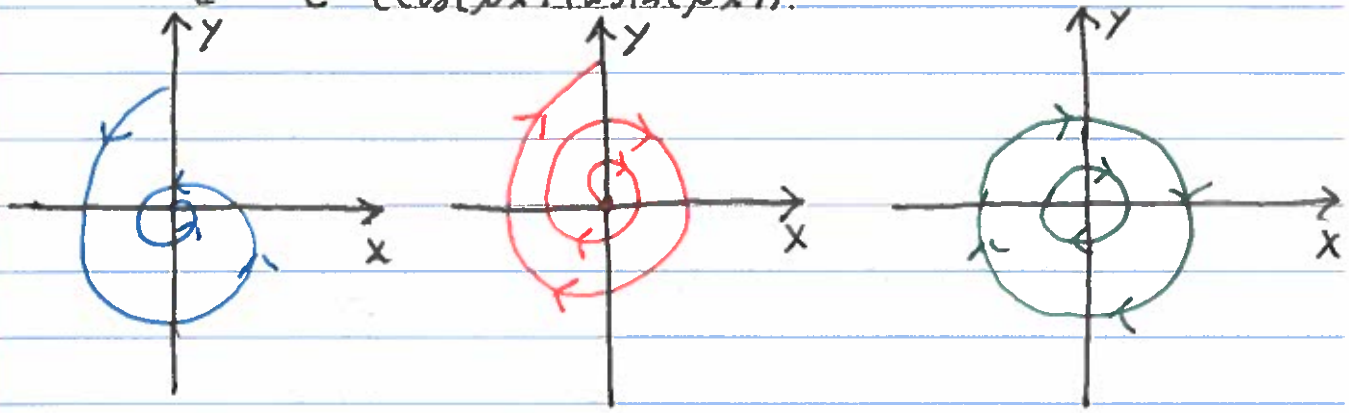
$\lambda_2<\lambda_1=0$ (Line of fixed points)

2. A has complex eigenvalues $\lambda, \bar{\lambda} \in \mathbb{C}\setminus\mathbb{R}$ with complex eigenvectors $\vec{V}_1, \vec{V}_2 \in \mathbb{C}$

$$x(t) = c_1 \, Re[e^{\lambda t}\vec{v}] + c_2 \, Im[e^{\lambda t}\vec{v}]$$

If we write $\lambda = \nu + i\mu$ then

$$e^{\lambda t} = e^{\nu t}(\cos(\mu t) + i\sin(\mu t)).$$



($Re(\lambda) < 0$) Stable Spiral

($Re(\lambda) > 0$) Unstable Spiral

($Re(\lambda) = 0$) Center

Terminology

1. A is called hyperbolic if $Re(\lambda_1), Re(\lambda_2) \neq 0$.
2. We say the fixed point $x = 0$ of $\dot{x} = Ax$ is
   (a) An attractor if $\vec{x}(t) \to 0$ as $t \to \infty \Rightarrow Re(\lambda_1, \lambda_2) < 0$,
   (b) A repellor if $\vec{x}(t) \to 0$ as $t \to -\infty \Rightarrow Re(\lambda_1, \lambda_2) > 0$,
   (c) A saddle if $\lambda_1 < 0 < \lambda_2$,
   (d) Nonhyperbolic if $Re(\lambda_1) = 0$ or $Re(\lambda_2) = 0$.