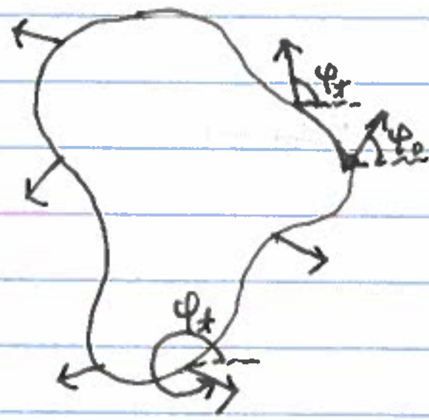
Lecture 17: Index Theory

How can we be sure no periodic orbit exists? Consider
$\dot{\vec{x}} = \vec{F}(\vec{x})$
with $F: \mathbb{R}^2 \to \mathbb{R}^2$ continuously differentiable.

Take a closed curve Γ with no self intersections, that does not pass through a fixed point.

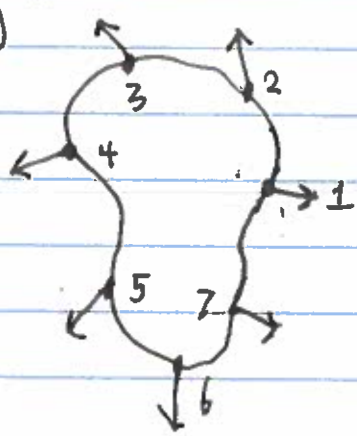


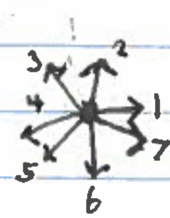
1. Start at x_0, transverse Γ ... counterclockwise and take angle φ of $F(\vec{x})$. This angle changes continuously as Γ is transversed.
2. After one pass we again end up at x_0 with an angle $\varphi_f = \varphi_0 + 2\pi n$.

Define: $I_\Gamma = \frac{1}{2\pi}(\varphi_f - \varphi_0)$, the index of a curve.

Examples:

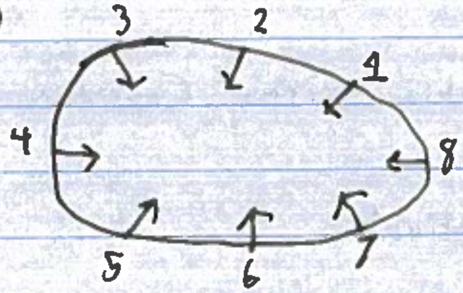
(a)

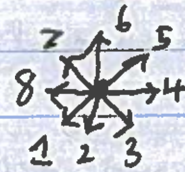




$\Rightarrow I_\Gamma = 1$. (Repeller)

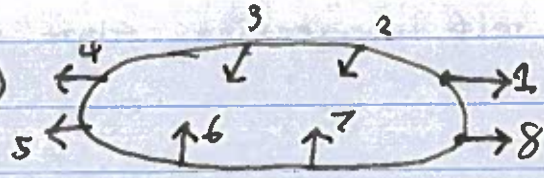
(b)

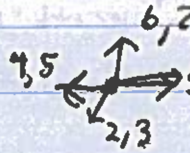




$\Rightarrow I_\Gamma = 1$ (Asymptotically stable fixed point)

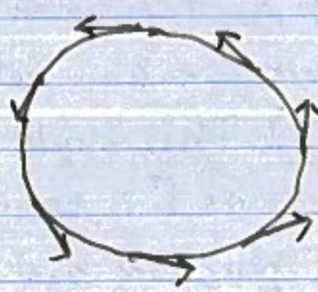
(c)





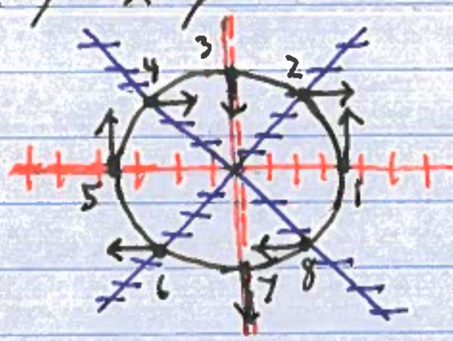
$\Rightarrow I_\Gamma = -1$ (Saddle)

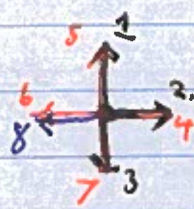
(d).



Periodic orbit $\Rightarrow I_\Gamma = 1$

(e). $\begin{cases} \dot{x} = x^2 y \\ \dot{y} = x^2 - y^2 \end{cases}$, Γ = unit circle





, $I_\Gamma = 0$ (Not a saddle!!)

Properties of the Index

1. If Γ can be deformed continuously into $\tilde{\Gamma}$ without passing through any equilibrium points then
$$I_\Gamma = I_{\tilde{\Gamma}}.$$
~~proof~~:
I_Γ varies continuously as Γ is deformed, but I_Γ is integer valued.

2. If Γ does not contain any fixed points then $I_\Gamma = 0$.
~~proof~~.
Property 1 implies we can shrink Γ to a point without changing the index.

3. If we replace $F(\vec{x})$ by $F(-\vec{x})$ the index is not changed.
~~proof~~:
Each angle is replaced by $\varphi + \pi$, hence $\varphi_f - \varphi_0$ is the same.

4. The index of a periodic orbit is 1.

Theorem - Assume F is continuously differentiable. Inside each periodic orbit, there is at least one periodic orbit.
~~proof~~:
Follows from items 2 and 4.

Index of isolated fixed point: Let $\vec{x}^*$ be an isolated fixed point of $\dot{\vec{x}} = F(\vec{x})$. Define
$I(\vec{x}^*)$ = index of simple closed curve that encloses $\vec{x}^*$ and no other fixed point.

* $I(\vec{x}^*)$ is well defined by property 1.

Consequences:

1. If $\vec{x}^*$ is an attractor or repellor then $I(\vec{x}^*) = 1$.
2. If $\vec{x}^*$ is a saddle then $I(\vec{x}^*) = -1$.

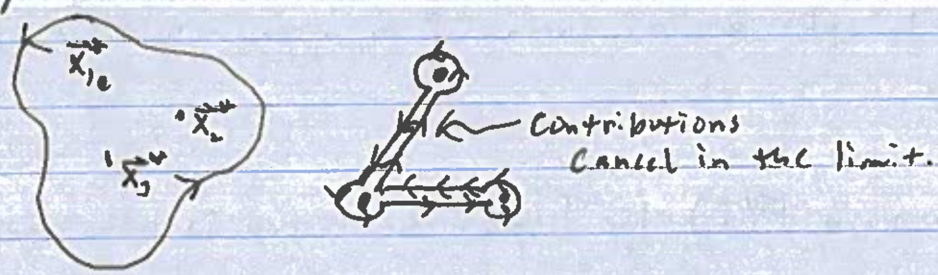
proof:

Follows from examples and properties 1, 3.

Theorem - If Γ is a closed simple curve that contains n isolated fixed points $\vec{x}_1^*, \dots, \vec{x}_n^*$ then

$$I_\Gamma = I(\vec{x}_1^*) + \dots + I(\vec{x}_n^*).$$

proof:



Corollary - A periodic orbit must enclose fixed points whose sum is 1.