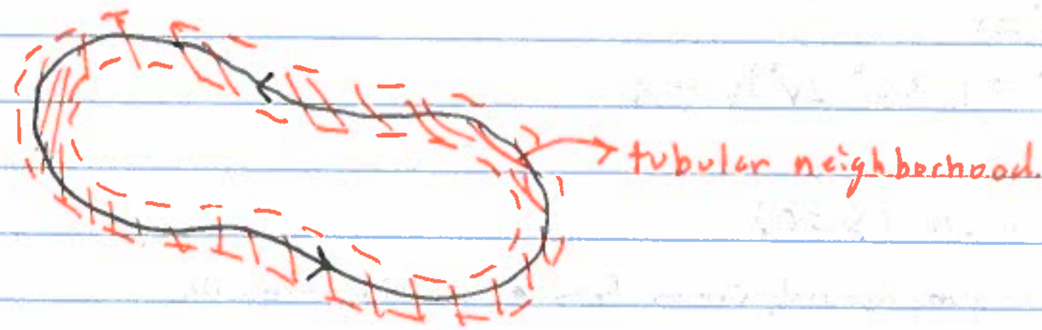
Lecture 19: Limit Cycles

Definition - A closed trajectory is a limit cycle if it is seperated from all other closed trajectories.

(a) A limit cycle is stable if there is a tubular neighborhood such that trajectories that enter the neighborhood approach the limit cycle as $t \to \infty$.

(b) A limit cycle is unstable if it is not stable

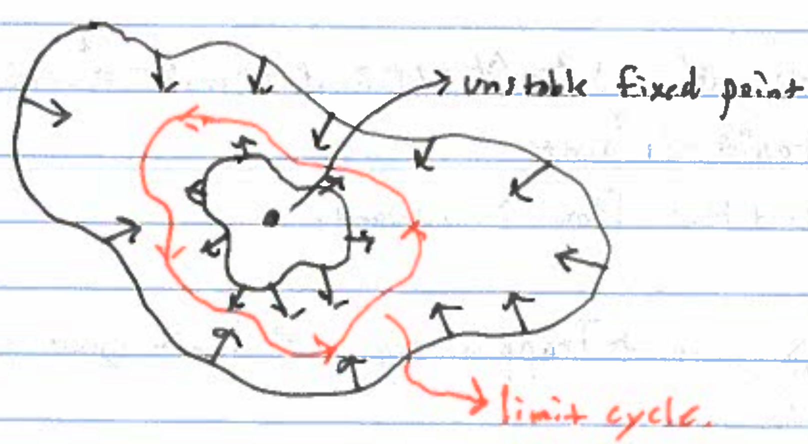


Theorem (Poincare Bendixon) - Consider $\dot{\vec{x}} = F(\vec{x})$ with F continuously differentiable. Assume $\mathcal{R} \subset \mathbb{R}^2$ is closed and bounded

(i) $\mathcal{R}$ does not contain any fixed points

(ii) For all $\vec{x}(0) \in \mathcal{R}$, $\vec{x}(t) \in \mathcal{R}$.

Then $\mathcal{R}$ contains a limit cycle.



Example:

$$\ddot{x} = (1-3x^2-2\dot{x}^2)\dot{x}^2 - x$$ (Models current in a vacuum tube).

Multiply by $\dot{x}$:

$$\frac{d}{dt}\underbrace{\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2\right)}_{\text{Energy of harmonic oscillator}} = \underbrace{(1-3x^2-2\dot{x}^2)\dot{x}^2}_{\text{Energy added or removed from system.}}$$

Write as a system:

$$\dot{x} = v$$
$$\dot{v} = (1-3x^2-2v^2)v - x$$

The nullclines are

<u>N1</u>: $v = 0$ $(\dot{x} = 0)$

<u>N2</u>: complicated, comes from quadratic formula.

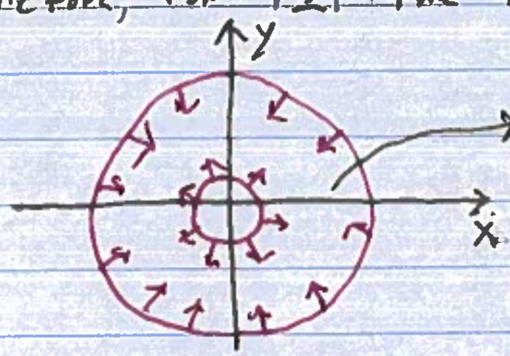
$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \lambda_{1,2} = \frac{1 \pm i\sqrt{3}}{2}$$ (unstable spiral)

We can convert to polar coordinates

$$r^2 = x^2 + y^2$$
$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r}$$

$$\dot{r} = r^2\cos\theta\sin\theta + r\sin\theta(1-3r^2\cos^2\theta - 2r^2\sin\theta)r\sin\theta - r^2\cos\theta\sin\theta$$
$$= r^2\sin^2\theta(1-3r^2\cos^2\theta - 2r^2\sin\theta)$$

Therefore, for $r \geq 1$ the flow is inward.



Trapping region $\Rightarrow$ limit cycle exists.

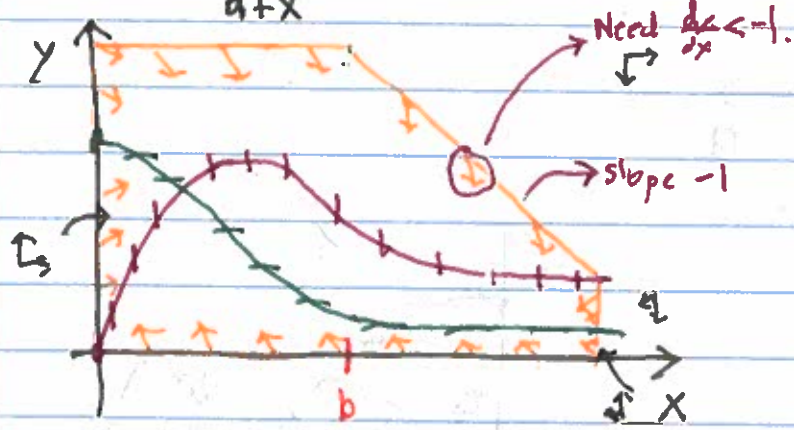
Example:

$$\dot{x} = -x + ay + x^2 y \quad (b, a > 0). \quad \text{(Chemical species } x, y\text{)}$$
$$\dot{y} = b - ay - x^2 y$$

Global Analysis:

N1: $y = \frac{x}{a+x^2}$ $(\dot{x}=0)$

N2: $y = \frac{b}{a+x^2}$ $(\dot{y}=0)$



$$\frac{dy}{dx} = \frac{b-ay-x^2y}{-x+ay+x^2y} < -1$$

$\Rightarrow b - \cancel{ay} - \cancel{x^2y} < x - \cancel{ay} - \cancel{x^2y}$

$\Rightarrow x > b$

Local Analysis

- Fixed point at $(b, \frac{a}{a+b^2})$.
- $J(x,y) = \begin{bmatrix} -1+2xy & a+x^2 \\ -2xy & -a-x^2 \end{bmatrix}$

$$J(b, \tfrac{a}{a+b^2}) = \begin{bmatrix} -1 + \frac{2ba}{a+b^2} & a+b^2 \\ -\frac{2ba}{a+b^2} & -a-b^2 \end{bmatrix}$$

$$Tr(J) = -1 - a - b^2 + \frac{2b^2}{a+b^2}, \quad det(J) = a+b^2 > 0.$$

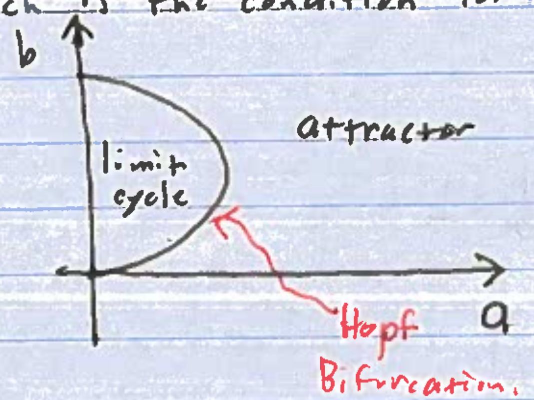
Recall,

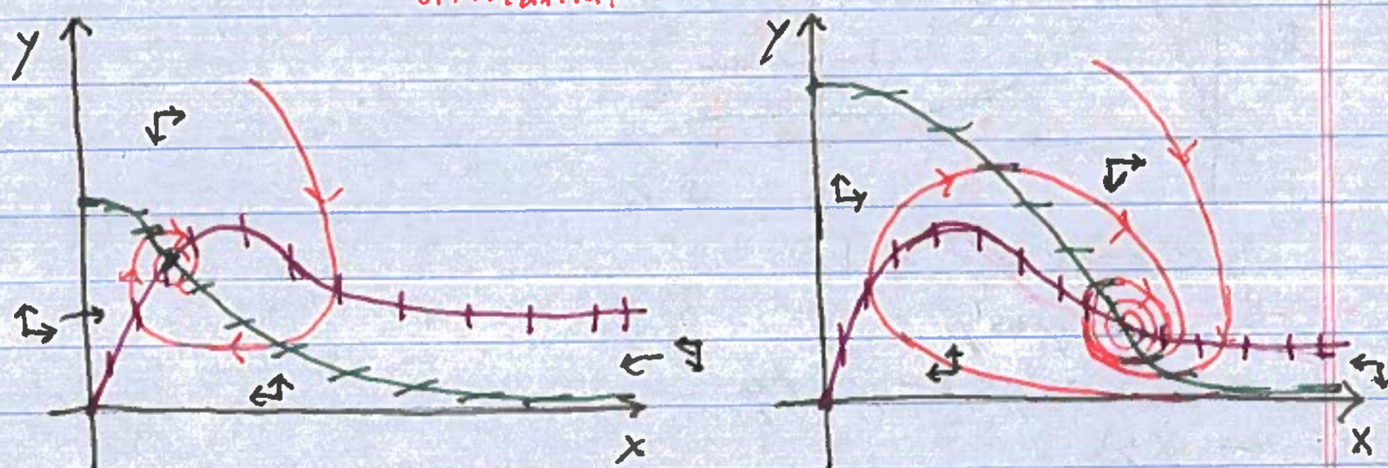
$$\lambda_{1,2} = Tr(J) \pm \sqrt{Tr(J)^2 - 4\det(J)}$$

In our case the fixed point is unstable if

$$Tr(J) > 0 \rightarrow -1 - a - b^2 + \frac{2b^2}{a+b^2} > 0.$$

Which is the condition for the existence of a limit cycle.





Van der Pol Oscillator

$$\ddot{x} + \varepsilon^{-1}(x^2-1)\dot{x} + x = 0, \quad \varepsilon << 1$$

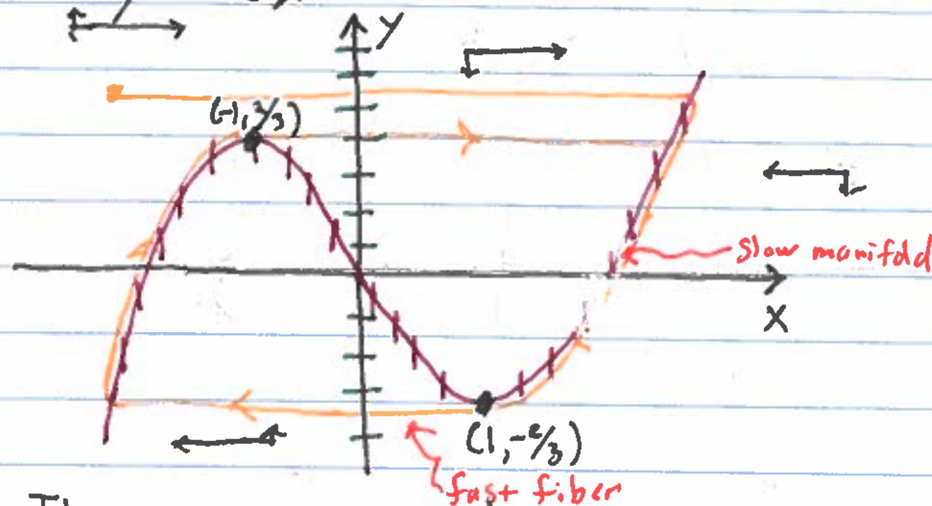
$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + V(x)\right) = -\frac{1}{\varepsilon}(x^2-1)\dot{x}^2 \leftarrow \text{like a pumping term.}$$

Now

$$\ddot{x} + \varepsilon^{-1}(x^2-1)\dot{x} = \frac{d}{dt}\left(\dot{x} + \varepsilon^{-1}(x^3/3 - x)\right)$$

Let $F(x) = x^3/3 - x$, $y = \varepsilon\dot{x} + x^3/3 - x$

$$\Rightarrow \dot{x} = \tfrac{1}{\varepsilon}(y - F(x)) \quad \dot{y} = -\varepsilon x \Bigg\} \text{ Fenichel transform.}$$



The motion is much larger in the x-direction. What is the shape of solution curves? What is the period of oscillations?

- On the fast fiber $y \approx$ constant

$$\Rightarrow \dot{x} = \tfrac{1}{\varepsilon}(y - F(x)) \quad \text{(Reduce to first order ODE).}$$
$$\approx \tfrac{1}{\varepsilon}\left(\tfrac{2}{3} - F(x)\right)$$

Let T_{fast} denote time on fast fiber.

$$\Rightarrow \int_{-1}^{1} \frac{1}{\frac{2}{3} - F(x)}\, dx = \frac{1}{\varepsilon} \int_0^{T_{fast}} dt$$

$$\Rightarrow T_{fast} = C_1 \cdot \varepsilon.$$

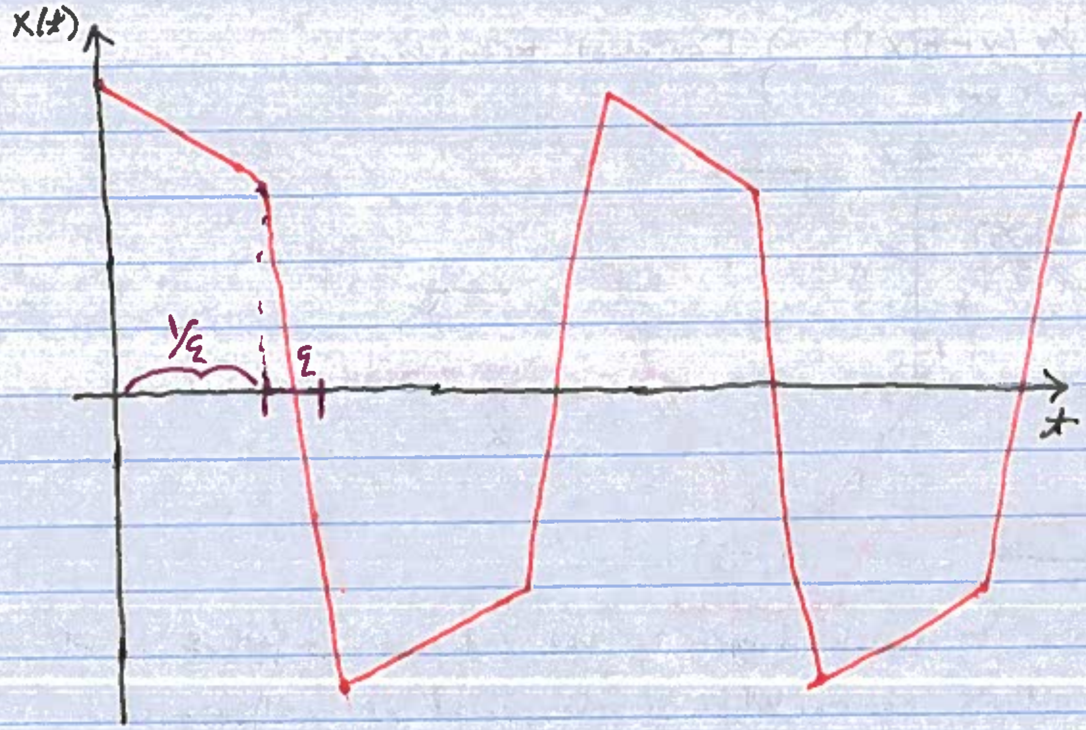
- On the slow manifold $y = x^3/3 - x$

$$\frac{dy}{dt} = (x^2 - 1)\dot{x}$$

$$\Rightarrow -\varepsilon x = (x^2 - 1)\dot{x}$$

$$\Rightarrow \int_{x_0}^{1} \frac{1 - x^2}{x}\, dx = \varepsilon \int_0^{T_{slow}} dt$$

$$T_{slow} = \frac{C_2}{\varepsilon}$$


x(t)
1/ε
ε
t