Lecture 20: Co-dimension one bifurcations

$$\dot{x} = f(x,y;\nu)$$
$$\dot{y} = g(x,y;\nu)$$

A bifurcation point ν^* is a point where the topology of the phase portrait changes.

- Recall, in 1-D

$$\dot{x} = f(x;\nu)$$

a (standard) bifurcation occurs at (x^*, ν^*) if

$$f(x^*;\nu^*) = 0$$
$$\left.\frac{df}{\partial x}\right|_{(x^*,\nu^*)} = 0$$

Near the bifurcation:

$$\dot{x} \approx f(x^*;\nu^*) + \left.\frac{\partial f}{\partial x}\right|_{(x^*,\nu^*)}(x-x^*) + \left.\frac{\partial f}{\partial \nu}\right|_{(x^*,\nu^*)}(\nu-\nu^*) + \frac{1}{2}\left(\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*,\nu^*)}(x-x^*)^2 + 2\left.\frac{\partial^2 f}{\partial x \partial \nu}\right|_{(x^*,\nu^*)}(x-x^*)(\nu-\nu^*) + \left.\frac{\partial^2 f}{\partial \nu^2}\right|_{(x^*,\nu^*)}(\nu-\nu^*)^2\right) + \dots$$

Lowest order:

$$\dot{x} \approx a(\nu-\nu^*) + b(x-x^*)^2 + c(x-x^*)(\nu-\nu^*) + d(\nu-\nu^*)^2$$

- $\dot{x} \approx a(\nu-\nu^*) + b(x-x^*)^2$ = saddle node bifurcation.
- $\dot{x} \approx b(x-x^*)^2 + c(x-x^*)(\nu-\nu^*)$ = transcritical
- $\dot{x} \approx$ cubic = pitchfork

- In 2-dimensions
  1. Let (x^*, y^*) denote the equilibrium point.
  2. Let λ_1, λ_2 denote eigenvalues associated with $J(x^*, y^*)$.

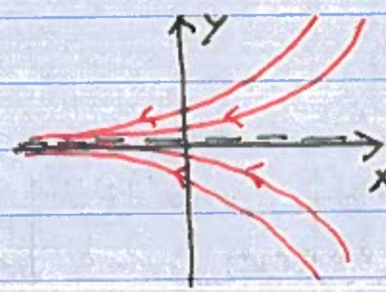
One type of bifurcation occurs if one or both eigenvalues lie on the imaginary axis.

- $\lambda_{1,2} = \pm i\omega \rightarrow$ Hopf bifurcation (new stuff)
- $\lambda_1 = 0,\ \lambda_2 \neq 0 \rightarrow$ 1-D bifurcations (old stuff)
- $\lambda_{1,2} = 0 \rightarrow J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow$ Takens Bogdanov (beyond the scope of course).

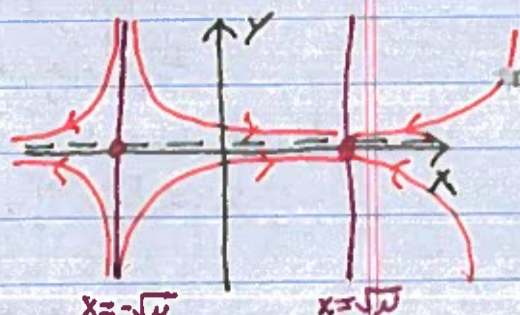
Examples:

1. Saddle Node:

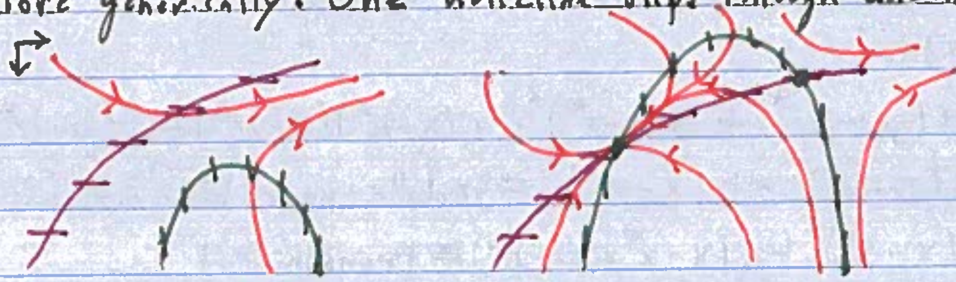
$$\dot{x} = \mu - x^2$$
$$\dot{y} = -y$$



$\mu < 0$



More generically: One nullcline slips through another



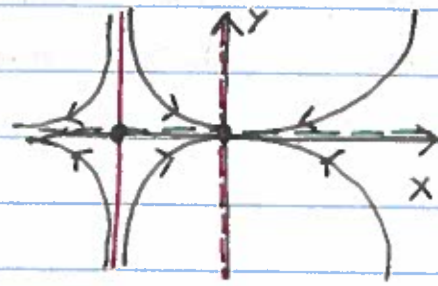
$\mu < \mu^*$ $\qquad$ $\mu > \mu^*$

2. **Transcritical:**

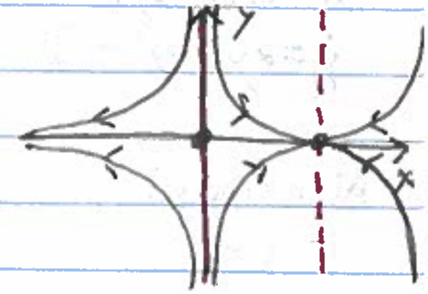
$\dot{x} = \mu x - x^2$

$\dot{y} = -y$

(nullcline passes through nullcline while preserving fixed point)



$\mu < 0$

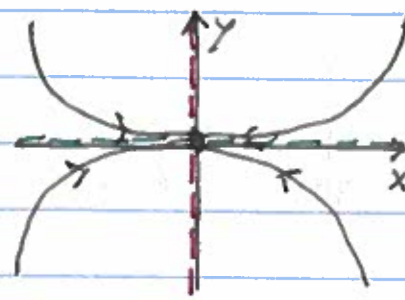


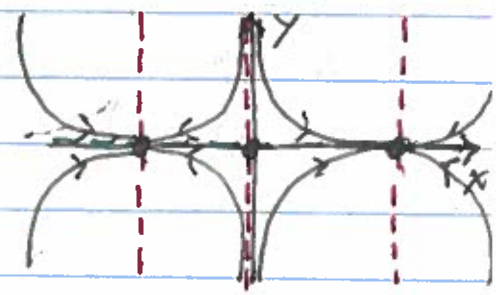
$\mu > 0$.

3. **Pitchfork:**

$\dot{x} = \mu x - x^3$

$\dot{y} = -y$





Example:

(spruce-budworm model)

$S \sim$ size of trees

$E \sim$ energy resources

$B \sim$ budworm population

$$\dot{S} = r_s S\left(1 - \frac{S}{K_s E / K_E}\right)$$

carrying capacity depends on energy

$$\dot{E} = \underbrace{r_E E\left(1 - \frac{E}{K_E}\right)}_{\text{logistic growth of energy}} - \underbrace{\frac{pB}{S}}_{\text{consumption of energy}}$$

rate of energy consumption of worms

budworm population

$x = S/K_s,\ y = E/K_E,\ \tau = r_s t$

$\dot{x} = x(1 - x/y)$
$\dot{y} = \alpha y(1-y) - \beta/x$

$\alpha = \frac{r_E}{r_s}$, $\beta = \frac{P B}{K_s r_s}$ → dimensionless measure of budworm population.

Nullclines:

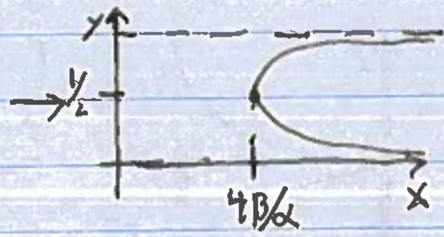
1. $\dot{x} = 0$:

   $x = 0$

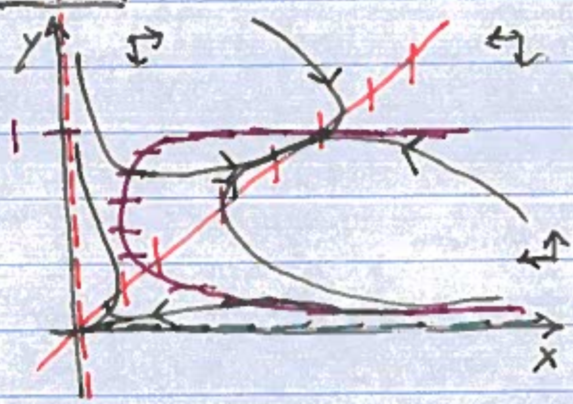
   $y = x$

2. $\dot{y} = 0$:

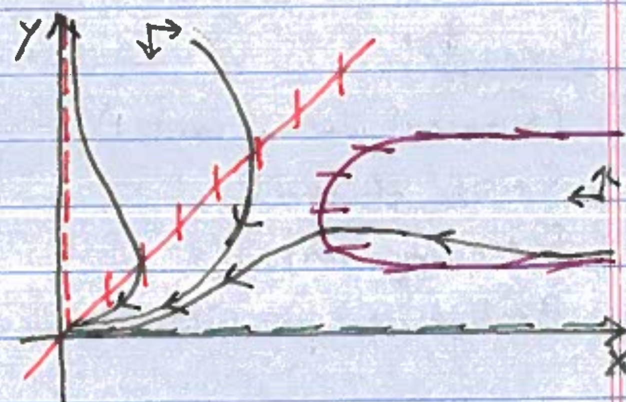
   $x = \frac{\beta}{\alpha y(1-y)}$ →



Cases:



Small β



Large β (extinction of forest)

Hopf-Bifurcation

$$\dot{x} = \mu x - \omega y \pm (x^3 + xy^2) + b(-x^2 y - y^3)$$
$$\dot{y} = \underbrace{\omega x + \mu y}_{\text{linear rotations}} \pm \underbrace{(x^2 y + y^3) + b(-x^3 + xy^2)}_{\text{cubic nonlinearity}}$$

$$J(0,0) = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} \Rightarrow \lambda_{1,2} = \mu \pm i\omega$$

Convert to polar coordinates:

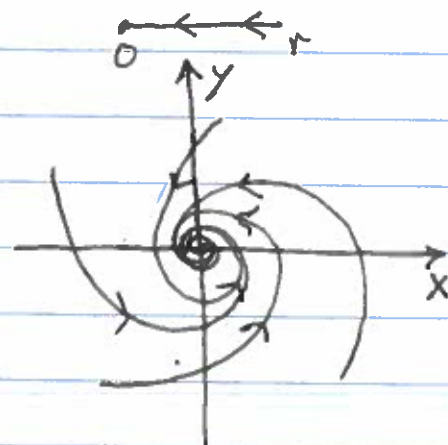
$$\dot{r} = \mu r \pm r^3$$
$$\dot{\theta} = \omega + b r^2$$

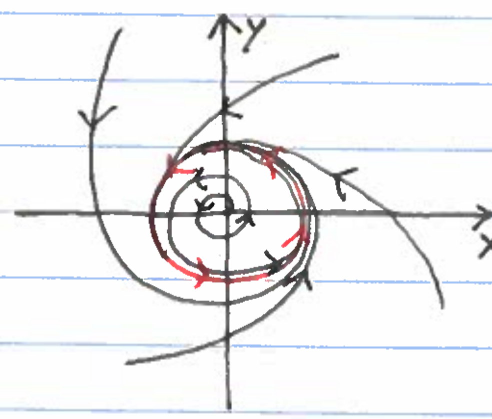
- Supercritical Hopf bifurcation for "−" sign

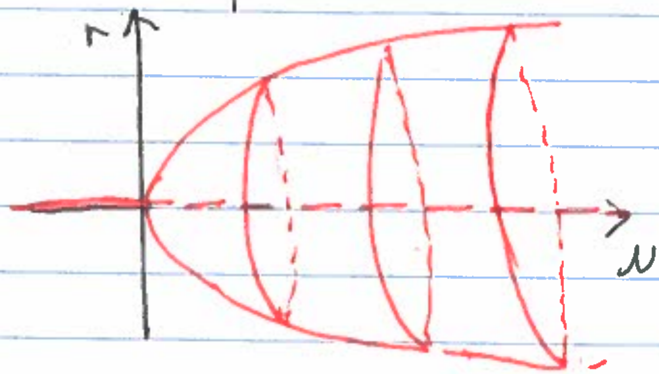
limit cycle $r = \sqrt{\mu}$

Case 1 ($\mu < 0$):

$$\dot{r} = \mu r - r^3$$







(Stable bifurcation born out of a stable fixed point)